VO Discrete Mathematics – VU Diskrete Mathematik

Exercises for Oct 12/13, 2011

1) A simple undirected graph is called cubic if each of its vertices has degree 3.

(a) Find a cubic graph with 6 vertices!

(b) Is there a cubic graph with an odd number of vertices?

(c) Prove that for all $n \geq 2$ there exists a cubic graph with $2n$ vertices!

2) Use a suitable graph theoretical model to solve the following problems:

(a) Show that in every city at least two of its inhabitants have the same number of neighbours!

(b) 7 friends want to send postcards according to the following rules: (i) Each person sends and receives exactly 3 cards. (ii) Each one receives only cards from those to whom he or she sent a card.

Tell how this can be done or prove that this is impossible!

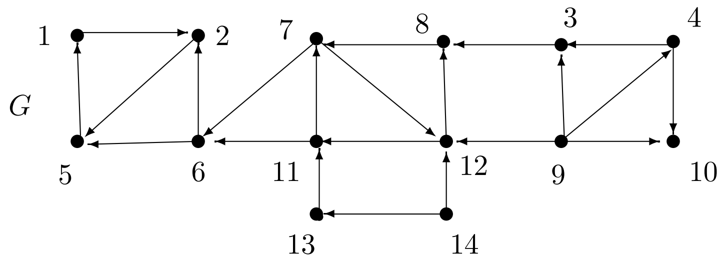
3) Show that each of the following statements is equivalent to the statement „T is a tree“:

1. Every two nodes of T are connected by exactly one path.
2. T is connected and $\alpha_0(T) = \alpha_1(T) + 1$.
3. T is a minimal connected graph, i.e., deleting an edge destroys connectivity.
4. T is a maximal acyclic graph, i.e., adding an edge generates a cycle.

4) Let $G = (V, E)$ be a simple and undirected graph with $|V| > 4$. The complement $G^\kappa = (V^\kappa, E^\kappa)$ of G is defined as follows: $V^\kappa = V$ and $vw \in E^\kappa$ if and only if $vw \notin E$. Show that either G or G^κ (or both) must contain a cycle! Furthermore, determine all trees T such that T^κ is a tree as well!

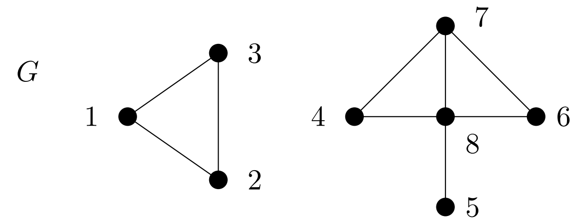
5) Let $G = (V, E)$ be a simple and undirected graph with $V = \{v_1, \ldots, v_n\}$. Its adjacency matrix is denoted by $A = (a_{ij})_{1 \leq i,j \leq n}$. Moreover, let $A^k = (a_{ij}^{[k]})_{1 \leq i,j \leq n}$ be the k-th power of A. Prove that $a_{ij}^{[k]}$ equals the number of walks from v_i to v_j having length k!

6) Find the strong connected components and the reduction G_R of the graph G below. Furthermore, determine all node bases of G.



7) Let $G = (V, E)$ be a simple and undirected graph and G_R its reduction. Prove that G_R is acyclic!

8) Use the matrix tree theorem to compute the number of spanning subgraphs of the graph below!



9) K_n denotes the complete graph with n vertices. Show that the number of spanning trees of K_n is n^{n-2}!

Hint: Use the matrix tree theorem and delete the first column and the first row of $D(K_n) - A(K_n)$. Then add all rows (except the first) to the first one and observe that all entries of the new first row are equal to 1. Use the new first row to transform the matrix in such a way that the submatrix built of the second to the last row and second to the last column is diagonal matrix.

10) If T is a tree having no vertex of degree 2, then T has more leaves than internal nodes. Prove this claim

(a) by induction,

(b) by considering the average degree and using the handshaking lemma.