Geometric Analysis: Problem Sheet 1

1. Show that $x : [0,1] \to \mathbb{R}^2$ given by $x(t) = (t, t\sin(1/t))$ for $0 < t \leq 1$ and $x(0) = (0,0)$ is a parametrization of a non-rectifiable curve.

2. Show that a curve $x : [0,1] \to \mathbb{R}^2$ with a Lipschitz parametrization is rectifiable.

3. Let $C = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leq 1, 0 \leq z \leq 1\} \subset \mathbb{R}^3$. The Schwarz lantern $C_{m,n}$ is a polyhedron inscribed to the unit cylinder C depending on parameters m, n: Take $m(n+1)$ vertices, a regular m-gon at each height k/n, but staggered so that the vertices at even levels are at angles $2\pi j/m$ while those at odd levels are at angles $\pi(2j+1)/m$. The polyhedron $C_{m,n}$ consists of $2mn$ congruent isosceles triangles and $2m$ triangles at the base and top of C.
   (i) Find the area of $C_{m,n}$ as a function of m, n.
   (ii) Show that any limiting area greater than or equal to 4π can be achieved in some limit of $m, n \to \infty$.

4. For an outer measure μ on $\mathbb{R}^n$, let $\mathcal{M}(\mu)$ be the class of subsets of $\mathbb{R}^n$ that are μ-measurable. Show that if μ and ν are outer measures on $\mathbb{R}^n$, then $\mu + \nu$ is an outer measure and
$$\mathcal{M}(\mu) \cap \mathcal{M}(\nu) \subseteq \mathcal{M}(\mu + \nu).$$

5. Let μ be an outer measure on $\mathbb{R}^n$ and $E_i \in \mathcal{M}(\mu)$. Show that
   (i) $E_i \subset E_{i+1}$ for all $i \in \mathbb{N} \Rightarrow \mu(\bigcup_{i\in\mathbb{N}} E_i) = \lim_{i\to\infty} \mu(E_i)$
   (ii) $E_{i+1} \subset E_i$ for all $i \in \mathbb{N}$ and $\mu(E_1) < \infty \Rightarrow \mu(\bigcap_{i\in\mathbb{N}} E_i) = \lim_{i\to\infty} \mu(E_i)$
   Does (ii) also hold without the assumption $\mu(E_1) < \infty$?

6. Discuss Vitali's example (showing that Lebesgue measure is not σ-additive on all subsets of $\mathbb{R}$).