Geometric Analysis: Problem Set 2

1. Let μ be a measure on $\mathbb{R}^n$ and $p \in [1, \infty)$. If $u \in L^p(\mathbb{R}^n, \mu)$ and $u \geq 0$, then
$$\int_{\mathbb{R}^n} |u|^p d\mu = p \int_0^\infty t^{p-1} \mu(\{u > t\})\, dt.$$
Here $\{u > t\} = \{x \in \mathbb{R}^n : u(x) > t\}$.
(Hint: Apply Fubini to $\mu \times \mathcal{L}^1$ and $f(x,t) = p\, t^{p-1}\ \mathbb{1}_{(0,|u(x)|)}(t)$.)

2. Let $n \geq 1$, $s \in (0, n)$, and $\delta \in (0, \infty]$. Show that $\mathcal{H}^s$ is a Borel measure on $\mathbb{R}^n$, and that $\mathcal{H}^s_\delta$ is *never* a Borel measure on $\mathbb{R}^n$.

3. If μ and ν are Borel regular measures on $\mathbb{R}^n$ such that $\mu = \nu$ on $\mathcal{B}(\mathbb{R}^n)$, then $\mu = \nu$ on $\mathcal{P}(\mathbb{R}^n)$.

4. Let $I = [a, b]$ be a line segment with endpoints $a, b \in \mathbb{R}^n$. Show that $\mathcal{H}^1(I) = |b - a|$.

5. Prove that the Hausdorff dimension of the Cantor set is $\alpha = \log 2 / \log 3$.

6. Show that a linear functional $L : C_c(\mathbb{R}^n; \mathbb{R}^m) \to \mathbb{R}$ is continuous if and only if it is bounded.
(Hint: To show the 'only if' part, argue by contradiction: there exist a compact set $K \subset \mathbb{R}^n$ and a sequence $\{\varphi_i\} \subset C_c(\mathbb{R}^n; \mathbb{R}^m)$ such that $|\varphi_i| \leq 1$, spt $\varphi_i \subset K$, and $\langle L, \varphi_i \rangle \geq i$. To conclude, consider $\psi_i = i^{-1/2} \varphi_i$.