Geometric Analysis: Problem Set 3

1. (Fundamental lemma of the Calculus of Variations). Let A be an open set in $\mathbb{R}^n$ and ν a vector Radon measure on $\mathbb{R}^n$ with values in $\mathbb{R}^m$. If
$$\int_{\mathbb{R}^n} \varphi \cdot d\nu = 0 \quad \forall\, \varphi \in C_c^\infty(A;\mathbb{R}^m),$$
then $|\nu|(A) = 0$. In particular, if $u \in L^1_{loc}(\mathbb{R}^n, \mathbb{R}^m)$ and
$$\int_{\mathbb{R}^n} \varphi(x) \cdot u(x)\, dx = 0 \quad \forall\, \varphi \in C_c^\infty(A;\mathbb{R}^m),$$
then $u = 0$ a.e. on A.
(Hint: Use ε-regularization.)

2. Let μ be a Radon measure on $\mathbb{R}^n$, let $r > 0$, and define the two functions $u, v : \mathbb{R}^n \to [0, \infty)$ as $u(x) = \mu(\bar{B}(x,r))$ and $v(x) = \mu(B(x,r))$ (where $B(x,r)$ is the open ball with center x and radius r). Show that u is upper semicontinuous and v is lower semicontinuous.
(Hint: Define μ_x by $\mu_x(E) = \mu(E + x)$ for $x \in \mathbb{R}^n$ and show that $\mu_x \overset{*}{\rightharpoonup} \mu_{x_0}$, when $x \to x_0$.)

3. Let μ_i, μ be vector Radon measures on $\mathbb{R}^n$. If $\mu_i \overset{*}{\rightharpoonup} \mu$ as $i \to \infty$ and $r_k \to \infty$ as $k \to \infty$, then
$$\lim_{i\to\infty} |\mu_i|(B_{r_k}) = |\mu|(B_{r_k}) \ \ \forall\, k \in \mathbb{N} \ \ \Rightarrow \ \ |\mu_i| \overset{*}{\rightharpoonup} |\mu|.$$

4. If $u \in L^1_{loc}(\mathbb{R}^n)$ and
$$\int_{\mathbb{R}^n} u\, \nabla\varphi = 0, \quad \forall\, \varphi \in C_c^\infty(\mathbb{R}^n),$$
then there exists $c \in \mathbb{R}$ such that $u = c$ a.e. on $\mathbb{R}^n$.

5. If E is a set of locally finite perimeter, then $\mathbb{R}^n \backslash E$ is a set of locally finite perimeter and
$$\mu_{\mathbb{R}^n\backslash E} = -\mu_E, \ \ P(E) = P(\mathbb{R}^n \backslash E),$$
where μ_E is the Gauss-Green measure of E.

6. If E and F are sets of locally finite perimeter, then $\mu_E = \mu_F$ on the bounded Borel sets of $\mathbb{R}^n$ if and only if $|E \triangle F| = 0$. When is $\mu_E = -\mu_F$ on the bounded Borel sets of $\mathbb{R}^n$?