Geometric Analysis: Problem Set 5

1. If $u \in C_c^\infty(\mathbb{R}^n)$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ a Borel function, then for $A \subset \mathbb{R}^n$ open,
$$\int_A \phi(x)\,|\nabla u|\,dx = \int_{\mathbb{R}} \int_{\{u=t\}} \phi(x)\,dH^{n-1}(x).$$

2. If $v \in C_c^1(\mathbb{R}^n)$, then there is a sequence (v_i) of piecewise affine and continuous functions with compact support such that $v_i \to v$ in $L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} |\nabla v_i| \to \int_{\mathbb{R}^n} |\nabla v|$.

3. Given $p > 0$, $n \geq 2$, and A an open set in $\mathbb{R}^n$, the p-Cheeger problem in A is the variational problem
$$c_p(A) = \inf\big\{\frac{P(E)}{|E|^p} : E \subset A\big\}.$$
A minimizer E with spt $\mu_E = \partial E$ is called a p-Cheeger set of A. Prove the following statements:
(i) If $p < (n-1)/n$, then $c_p(A) = 0$.
(ii) If $p > (n-1)/n$ and A is bounded, then p-Cheeger sets exist.
(iii) If $p = (n-1)/n$, then balls contained in A are the only p-Cheeger sets in A.

4. If $f \in C_c^\infty(\mathbb{R}^n)$, then
$$\int_{\mathbb{R}^n} |\nabla f(x)|\,dx \geq n\,\omega_n^{1/n}\Big(\int_{\mathbb{R}^n} |f(x)|^{\frac{n}{n-1}}dx\Big)^{\frac{n-1}{n}}. \tag{1}$$
(Hint: Use the Minkowski inequality for integrals
$$\Big(\int\big(\int g(x,y)\,dy\big)^p dx\Big)^{1/p} \leq \int\Big(\int g(x,y)^p\,dx\Big)^{1/p} dy$$
for $p \geq 1$ (cf. Hardy, Littlewood & Pólya: Inequalities (6.13.9)), the coarea formula and the isoperimetric inequality).

5. Prove the isoperimetric inequality for bounded sets of finite perimeter in $\mathbb{R}^n$ by using the Sobolev inequality (1) and suitable approximations.