Geometric Analysis: Problem Set 6

1. Let μ be a Radon measure on $\mathbb{R}^n$. If $\{u_i\}_{i\in\mathbb{N}} \subset L^p(\mathbb{R}^n, \mu)$, where $1 < p \le \infty$, satisfies

$$\sup_{i\in\mathbb{N}} \|u_i\|_{L^p(\mathbb{R}^n,\mu)} < \infty,$$

then there exist a sequence $i(k) \to \infty$ as $k \to \infty$ and $u \in L^p(\mathbb{R}^n, \mu)$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \varphi\, u_{i(k)}\, d\mu = \int_{\mathbb{R}^n} \varphi\, u\, d\mu$$

for every $\varphi \in L^{p'}(\mathbb{R}^n, \mu)$, where $p' = 1$ for $p = \infty$ and $p' = p/(p-1)$ for $1 < p < \infty$.

Hint: By the compactness criterion for signed Radon measures, there exists a signed Radon measure ν such that $\mu_{i(k)} = u_{i(k)}\,\mu \overset{*}{\rightharpoonup} \nu$. Show that ν is absolutely continuous w.r.t. μ and that, in fact, $\nu = u\,\mu$ for $u \in L^p(\mathbb{R}^n, \mu)$.

2. Show that if $u \in L^1_{loc}(\mathbb{R}^n)$, then $u_\varepsilon \in C^\infty(\mathbb{R}^n)$, spt $u_\varepsilon \subset$ spt $u + \varepsilon\, B$,

$$\nabla u_\varepsilon(x) = \int_{\mathbb{R}^n} \nabla\rho_\varepsilon(x-y)\, u(y)\, dy, \quad \text{for} \quad x \in \mathbb{R}^n,$$

and $u_\varepsilon(E)$ is contained in the closed convex hull of $u(E)$ for every $E \subset \mathbb{R}^n$. Moreover, show that $u_\varepsilon \to u$ in $L^1_{loc}(\mathbb{R}^n)$, $u_\varepsilon(x) \to u(x)$ at Lebesgue points x of u, and, for $0 < R < \infty$,

$$\|u_\varepsilon\|_{L^1(B_R)} \le \|u\|_{L^1(B_{R+\varepsilon})}.$$

3. Let $x \in \mathbb{R}^n$, $r_0 > 0$ and $\{C_r\}_{0<r<r_0}$ be a family of Borel sets such that, for some $\beta > \alpha > 0$, $B(x, \alpha\, r) \subset C_r \subset B(x, \beta\, r)$ for $0 < r < r_0$ (for example, $C_r = x + r\,A$, A open and bounded, $0 \in A$). Show that if $E \subset \mathbb{R}^n$ is a Lebesgue measurable set and $t \in \{0, 1\}$, then $x \in E^{(t)}$ if and only if

$$\lim_{r\to 0+} \frac{|E \cap C_r|}{|C_r|} = t.$$

Also show that the equivalence does not hold for $0 < t < 1$.

4. Let the function $u \in L^1_{loc}(\mathbb{R}^n)$, the set $A \subset \mathbb{R}^n$ be open and connected, and let

$$\int_{\mathbb{R}^n} u\, \nabla\varphi = 0 \quad \text{for all} \quad \varphi \in C^\infty_c(A).$$

Show that there is a constant $c \in \mathbb{R}$ such that $u = c$ a.e. on A.