Differential geometry (104.358)
Exercise sheet for 03.5.2018

23. Calculate the induced metric of the catenoid

$$X : (u, v) \mapsto X(u, v) := O + e_1 \cosh(u) \cos(v) + e_2 \cosh(u) \sin(v) + e_3 u.$$

24. Show that a surface $X : \mathbb{R}^2 \supseteq U \to \mathcal{E}^3$ is conformally parametrised if and only if the parametrisation preserves angles.

25. Consider the sphere $S^2 \subset \mathbb{R}^3$ with radius 1 and centred at the origin. Let $N = (0, 0, 1) \in S^2$ be the north pole of the sphere. For every point $(u, v, 0)$ in the e_1e_2-plane draw a line through $(u, v, 0)$ and N. This line cuts the sphere in the north pole and another point $\Pi(u, v)$ and thus defines a map

$$\begin{aligned} \Pi : \mathbb{R}^2 &\to S^2 \subset \mathbb{R}^3, \\ (u, v) &\mapsto \Pi(u, v). \end{aligned}$$

Calculate the map Π. What is the image of Π? What is $\lim_{u\to\infty} \Pi(u, v)$?

After a choice of origin $O \in \mathcal{E}^3$, define Π a map X to $\mathcal{E}^3$,

$$\begin{aligned} X : \mathbb{R}^2 &\to \mathcal{E}^3, \\ (u, v) &\mapsto O + \Pi(u, v). \end{aligned}$$

Convince yourself that X is a regular parametrisation of the sphere without the north pole. Is the parametrisation conformal?

What is the image of a line in $\mathbb{R}^2$ under X? What is the image of a circle in $\mathbb{R}^2$ with centre at the origin under X?

Note: the inverse function Π^{-1} is called stereographic projection.

26. Let $r > 0$ and define a parametrisation of the Möbius strip by

$$\begin{aligned} X : \mathbb{R}^2 &\to \mathcal{E}^3, \\ (u, v) &\mapsto O + (e_1 \cos 2u + e_2 \sin 2u)(r + v \cos u) + e_3 v \sin u. \end{aligned}$$

Calculate the Gauss map N of X.

Prove that $X(u + \pi) = X(u, 0)$ but $N(u + \pi) = -N(u, 0)$.

27. Let

$$\begin{aligned} X :\ \mathbb{R}^2 \supseteq U &\to \mathcal{E}^3, \\ (u, v) &\mapsto O + e_1 u + e_2 v + e_3 z(u, v), \end{aligned}$$

where $z :\ U \to \mathbb{R}$ is smooth.

Calculate the Gauss map and the shape operator of X.

Now let (u_0, v_0) be a stationary point of the function z. What does the shape operator and the Gauss curvature look like at this point?