Differential geometry (104.358)
Exercise sheet for 07.6.2018

38. Prove: In curvature line coordinates (u, v) the Codazzi equations have the form

$$0 = \kappa_{1v} + \frac{E_v}{2E}(\kappa_1 - \kappa_2) = \kappa_{2u} - \frac{G_u}{2G}(\kappa_1 - \kappa_2).$$

39. Suppose that a surface $O + e_1x + e_2y + e_3z$ is implicitly defined through $F(x, y, z) = 0$ where $F : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function satisfying $\operatorname{grad} F \neq 0$ wherever $F(x, y, z) = 0$. A parametrisation $X : (u, v) \mapsto X(u, v)$ of this surface therefore satisfies $F \circ (X - O) = 0$. Let $C : t \mapsto C(t)$ be a curve on this surface, i.e., $F \circ (C - O) = 0$.

    Show that the natural ribbon of C is given by (C, N) where

$$N = \pm \frac{\operatorname{grad} F \circ (C - O)}{|\operatorname{grad} F \circ (C - O)|}.$$

    Use this to prove that the two curves

$$C_\pm : t \mapsto C_\pm(t) = O + e_1 + e_2t \pm e_3t$$

    on the one-sheeted Hyperboloid

$$\{O + e_1x + e_2y + e_3z \in \mathcal{E}^3 \mid x^2 + y^2 - z^2 - 1 = 0\}$$

    are asymptotic and pre-geodesic lines, but not curvature lines.

40. Prove Joachimsthal's theorem:

    Suppose that two surfaces intersect along a curve and that the curve is a curvature line for one of the two surfaces; then it is a curvature line for the other surface if and only if the two surfaces intersect at a constant angle.

41. Find a curvature line reparametrisation of the helicoid:

$$X(u, v) = O + e_1 \sinh u \cos v + e_2 \sinh u \sin v + e_3hv.$$

42. Prove Euler's theorem:

    The normal curvatures κ_n at a point of a surface satisfy

$$\kappa_n = \kappa^+ \cos^2\theta + \kappa^- \sin^2\theta,$$

    where $\kappa^\pm$ are the principal curvatures and θ is the angle between the tangent direction of $\kappa_n(\theta)$ and the curvature direction of κ^+.

    Hint: fix $(u, v) \in M$ and use a basis (e_1, e_2) of $\mathbb{R}^2$ that is orthonormal for $\mathrm{I}|_{(u,v)}$ and diagonalises $\mathrm{II}|_{(u,v)}$.

43. Fix a point $X(u, v)$ on a parametrised surface and assume that this is not a flat point. Prove that an asymptotic line can pass through $X(u, v)$ in two, one or no directions depending on the sign of the Gauss curvature $K(u, v)$.