Differential geometry (104.358)
Exercise sheet for 12.4.2018

9. Let (X, N) be a ribbon for the curve $X : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^3$ and let $\tilde{N} := N \cos\phi + B \sin\phi$, where $\phi : \mathcal{I} \to \mathbb{R}$ is differentiable and $B = T \times N$.

   Show that $(X, \tilde{N})$ is a ribbon and calculate how the curvatures κ_n, κ_g and the torsion τ change when switching from (X, N) to $(X, \tilde{N})$.

10. Let (X, N) be a ribbon for the curve $X : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^3$ and let $\tilde{X} := X \circ \psi$ be a reparametrisation of X, i.e., $\psi : \mathcal{I} \to \mathcal{I}$ is a bijective differentiable function with $\psi' \neq 0$.

    (a) Show that $(\tilde{X}, \tilde{N})$ with $\tilde{N} = N \circ \psi$ is a ribbon.

    (b) How does the curvatures κ_n, κ_g and torsion τ change between (X, N) and $(\tilde{X}, \tilde{N})$?

11. Let $X : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^3$ be a regular parametrisation of a line, i.e., $X' \times X'' = 0$, and let F be a suitable frame for X.

    Show that $\kappa_n = 0 = \kappa_g$ and find a unit normal field N so that $\tau = 1$.

12. Consider the parametrised curve

    $$X : \mathbb{R} \to \mathcal{E}^3,$$
    $$t \mapsto O + e_1 t + e_2 t^2 + e_3 \frac{2t^3}{3}.$$

    Calculate the arc length of X.

    Find a normal field N for X and calculate at $t = 0$ the normal curvature, geodesic curvature and torsion by using the identities

    $$\kappa_n = -(N', T), \qquad \kappa_g = -(T', B), \qquad \tau = (N', B).$$

13. Prove:

    An arc length parametrised curve X in $\mathcal{E}^3$ lies in a plane if and only if there exists a suitable frame such that $\kappa_g = 0 = \tau$.

    Is the same statement true for $\kappa_n = 0 = \tau$ and for $\kappa_n = 0 = \kappa_g$?