Differential geometry (104.358)
Exercise sheet for 17.5.2018

27. Calculate the Gauss map and shape operator of the catenoid

$$X : (u, v) \mapsto X(u, v) := O + e_1 \cosh u \cos v + e_2 \cosh u \sin v + e_3 u.$$

Show that the catenoid is a minimal surface, i.e., the mean curvature H is zero.

28. Assume a surface

$$\begin{aligned} X :&\mathbb{R}^2 \supseteq U \to \mathcal{E}^3, \\ &(u, v) \mapsto X(u, v) = O + e_1 x + e_2 y + e_3 z, \end{aligned}$$

is implicitly defined through an equation $F(x, y, z) = 0$ with a smooth map $F : \mathbb{R}^3 \to \mathbb{R}$ with $\operatorname{grad} F \neq 0$ whenever $F(x, y, z) = 0$. We can equivalently define the surface through the equation $F \circ (X - O) = 0$. Show that the Gauss map is given by

$$N = \pm \frac{(\operatorname{grad} F) \circ (X - O)}{\|(\operatorname{grad} F) \circ (X - O)\|}$$

(grad denotes the gradient, i.e., the vector of partial derivatives).

29. Calculate the Gauss map and the shape operator of the helicoid:

$$(r, v) \mapsto X(r, v) = O + e_1 r \cos v + e_2 r \sin v + e_3 v.$$

Determine also the mean curvature, the two principal curvatures and the curvature directions of X.

30. Investigate how the first and second fundamental forms of a surface change under reparametrisation and Euclidean motions.

31. Show that the circular cone with its apex removed,

$$\{O + e_1 x + e_2 y + e_3 z \mid x^2 + y^2 = z^2,\ z > 0\},$$

admits an isometric parametrisation.

Hint: Find a simple parametrisation and then do an appropriate reparametrisation of the domain.