Differential geometry (104.358)
Exercise sheet for 19.4.2018

14. (a) Let $Y : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^2 \subset \mathcal{E}^3$ be a plane curve. Then $\frac{Y'(t)}{\|Y'(t)\|} = e_1 \cos(\psi(t)) + e_2 \sin(\psi(t))$ for some function $\psi : \mathcal{I} \to \mathbb{R}$.
Convince yourself that
$$\kappa(t) := \frac{\det\left(Y'(t), Y''(t)\right)}{\|Y'(t)\|^3} = \frac{\psi'(t)}{\|Y'(t)\|},$$
and thus κ measures the change in direction of the tangent vector.

    (b) Let $X : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^3$ be an arc length parametrised curve in $\mathcal{E}^3$, let N be any normal field and fix $t_0 \in \mathcal{I}$. Project X orthogonally to the affine plane through $X(t_0)$, that is spanned by $X'(t_0)$ und $N(t_0)$. Identify the plane with $\mathcal{E}^2$ to obtain a planar curve $\tilde{X} : \mathcal{I} \to \mathcal{E}^2$, the orthogonal projection of X.
Show that
$$|\kappa(t_0)| = \left| \frac{\det\left(\tilde{X}'(t_0), \tilde{X}''(t_0)\right)}{\|\tilde{X}'(t_0)\|^3} \right| = |\langle T'(t_0), N(t_0)\rangle| = |\kappa_n(t_0)|.$$

15. Let $X : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^3$ be a plane curve and N a unit normal field that is tangent to this plane. Show that N is parallel for the normal connection. What do the other parallel normal fields along X look like?

16. Let $X : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^3$ be a curve in $\mathcal{E}^3$. Prove:

    (a) Any two parallel frames of X differ by a constant rotation in the normal plane.

    (b) Let N be a parallel normal field along X. Then a normal field $\tilde{N}$ is parallel if and only if it makes a constant angle with N.

17. Show that a curve in $\mathcal{E}^3$ lies on a sphere or plane if and only if the curvatures κ_g and κ_n of a parallel frame satisfy a linear equation in $\mathbb{R}^2$. How can the radius of the sphere be read from this equation?

18. Let
$$\phi(t) = \begin{pmatrix} 0 & -\kappa_n(t) & \kappa_g(t) \\ \kappa_n(t) & 0 & -\tau(t) \\ -\kappa_g(t) & \tau(t) & 0 \end{pmatrix},$$
where $\kappa_n, \kappa_g, \tau : \mathbb{R} \supset \mathcal{I} \to \mathbb{R}$.
Show that
$$F(t) := \sum_{i=0}^{\infty} F_i(t) = \lim_{k\to\infty} \sum_{i=0}^{k} F_i(t)$$
with $F_0(t) = id_{\mathbb{R}^3}$ and
$$F_i(t) = \int_{t_0}^{t} F_{i-1}(\tilde{t})\phi(\tilde{t})\,\mathrm{d}\,\tilde{t} \qquad \text{for } i > 0$$
is a suitable frame for an arc length parametrised curve X, whose curvatures and torsion are κ_n, κ_g and τ. You may assume that $F(t)$ and $F'(t)$ exist and that
$$F'(t) = \sum_{i=0}^{\infty} F'_i(t)$$
holds. How does F simplify if ϕ is constant?