Differential Geometry (104.358)
Exercise sheet for 22.3.2018

4. Consider again the following curves:

   (a) Neile's semicubical parabola, $p:\ \mathbb{R} \to \mathcal{E}^3,\ t \mapsto O + e_1 t^3 + e_2 t^2$.

   (b) The graph of the absolute value function, $b:\ \mathbb{R} \to \mathcal{E}^3,\ t \mapsto O + e_1 t + e_2 |t|$.

   (c) The straight line, $g:\ \mathbb{R} \to \mathcal{E}^3,\ t \mapsto O + e_1 t^3$.

   For which of the three maps is there a diffeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ so that $\gamma \circ \psi : \mathbb{R} \to \mathcal{E}^3$, $\gamma \in \{p, b, g\}$, is regular?

   For which of the three maps is there a bijective, differentiable map $\phi : \mathbb{R} \to \mathbb{R}$ so that $\tilde{X} = X \circ \left(\phi^{-1}\right) : \mathbb{R} \to \mathcal{E}^3$, $X \in \{p, b, g\}$, is regular? (And for which is there none?)

5. Find a regular parametrisation for the conic section

   $$C_{\alpha,d} = \{O + e_1 x + e_2 y + e_3 z \in \mathcal{E}^3 \mid x^2 + y^2 = z^2,\ x \cos\alpha + z \sin\alpha = d\},$$

   $\alpha \in [0, \frac{\pi}{2}]$, $d \neq 0$.

   <u>Hint:</u> Separate the cases $\alpha < \frac{\pi}{4}$, $\alpha = \frac{\pi}{4}$ und $\alpha > \frac{\pi}{4}$.

6. Prove: A map

   $$\begin{aligned} X:\ \mathbb{R} \supseteq \mathcal{I} &\to \mathcal{E}^n, \\ t &\mapsto X(t), \end{aligned}$$

   with $X' \neq 0$ parametrises a line or a line segment if $X'(t)$ and $X''(t)$ are linearly dependent. In this case, what sort of reparametrisations $\tilde{X}(\tilde{t})$ of $X(t)$ satisfy $\tilde{X}'' = 0$?

7. Convince yourself that the arc length

   $$s(t) = \int_{t_0}^{t} \|X'(\tilde{t})\| \,\mathrm{d}\,\tilde{t}$$

   of a curve $t \mapsto X(t)$ is invariant under reparametrisation.

8. Consider the curve $X(t) = O + e_1 x(t) + e_2 y(t) + e_3 z(t)$ defined implicitly by the equations

   $$\left(\frac{x(t)}{a}\right)^2 + \left(\frac{y(t)}{b}\right)^2 + \left(\frac{z(t)}{c}\right)^2 = 1, \qquad a\sqrt{b^2 - c^2}\ z(t) = c\sqrt{a^2 - b^2}\ x(t),$$

   with $a > b > c > 0$.

   (a) Which two surfaces intersect to give the curve?

   (b) Find a parametrisation of the curve.

   (c) Calculate the arc length and an arc length parametrisation of the curve.

   (d) What does the curve look like?