Differential geometry (104.358)
Exercise sheet for 26.4.2018

18. (a) Convince yourself that the Frenet condition $\forall t \in \mathcal{I}: \; X'(t) \times X''(t) \neq 0$ of a curve $X : \mathcal{I} \to \mathcal{E}^3$ is invariant under reparametrisation.

    (b) Let $X : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^3$ be a Frenet curve. Show that (X, N) is a geodesic ribbon ($\kappa_g = 0$) if and only if $\pm N$ is the principal normal field.

19. Let $X : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^3$ be an arc length parametrised Frenet curve. The *Darboux vector field* is given by

    $$D := \tau T + \kappa B.$$

    Show that the Frenet equations can be written as

    $$T' = D \times T, \qquad N' = D \times N, \qquad B' = D \times B.$$

20. Let $\gamma : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^3$ be a Frenet curve. Show that (T, N, B) with

    $$T = \frac{\gamma'}{\|\gamma'\|}, \quad B = \frac{\gamma' \times \gamma''}{\|\gamma' \times \gamma''\|}, \quad N = B \times T$$

    is the Frenet frame of the curve.

21. Express the curvature and torsion of a Frenet curve in terms of κ_n and κ_g of a parallel frame and vice versa.

22. Show that the ellipsoid with two points removed

    $$E = \left\{ O + e_1 x + e_2 y + e_3 z \; \in \mathcal{E}^3 \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1, \; |z| < c \right\}, \qquad a > b > c,$$

    is a surface by finding a regular parametrisation.

23. Show that the torus

    $$T^2 := \left\{ O + e_1 x + e_2 y + e_3 z \; \in \mathcal{E}^3 \mid \left(\sqrt{x^2 + y^2} - R\right)^2 + z^2 = r^2 \right\}$$

    with $0 < r < R$ is a surface.

    Hint: How is the torus made up of circles? Use parametrisations of circles to find a parametrisation of the torus.