Differential geometry (104.358)
Exercise sheet for 28.6.2018

54. Prove the chain rule for differentiable maps between submanifolds, i.e., for submanifolds $M \subset \mathcal{E}^m$, $N \subset \mathcal{E}^n$ and $R \subset \mathcal{E}^r$, and differentiable maps $\phi : M \to N$ and $\psi : N \to R$ show that
$$d_p(\psi \circ \phi) = d_{\phi(p)}(\psi) \circ d_p \phi$$
for all $p \in M$.

55. The method of Lagrange-Multipliers:

    Let $M = F^{-1}(\{0\})$ be a $(n-1)$-dimensional submanifold of $\mathcal{E}^n$ for some submersion $F : \mathcal{E}^n \to \mathbb{R}$ and let $\Phi : \mathcal{E}^n \to \mathbb{R}$ a differentiable function. Show that $p \in M$ is a critical point (so possibly an extremum) of $\phi := \Phi\big|_M : M \to \mathbb{R}$ if and only if there exists $\lambda \in \mathbb{R}$ so that (λ, p) is a critical point of the function
$$\psi : \mathbb{R} \times \mathcal{E}^n \to \mathbb{R}, \quad (\lambda, p) \mapsto \Phi(p) - \lambda F(p).$$

56. Let $M = \mathcal{E}^2$ and define a parametrisation $X : \mathbb{R}^2 \to \mathcal{E}^2$, $X(u,v) = O + e_1 u + e_2 v$. Consider the vector fields $\xi, \eta : M \to \mathbb{R}^2$ defined by
$$\xi \circ X = e_1 v - e_2 u \quad \text{and} \quad \eta \circ X = e_1 u^2 + e_2 v.$$
Calculate the maximal flows of ξ and η.

57. Let $M \subset \mathcal{E}^3$ be an affine 2-plane in $\mathcal{E}^3$.
    (a) Find two tangent vector fields ξ, η so that $[\xi, \eta] = 0$ but $\nabla_\xi \eta \neq 0$.
    (b) Find two tangent vector fields ξ, η so that $[\xi, \eta] \neq 0$ but $\nabla_\xi \eta = 0$.

58. Suppose that $M \subset \mathcal{E}^m$ and $N \subset \mathcal{E}^n$ are submanifolds. Show that $M \times N$ is a submanifold of $\mathcal{E}^m \times \mathcal{E}^n$.

    Recall that $Gl(n) = \{A \in \mathrm{End}(\mathbb{R}^n) : \det A \neq 0\}$ is a n^2-dimensional submanifold of $\mathrm{End}(\mathbb{R}^n)$. Show that the multiplication map
$$Gl(n) \times Gl(n) \to Gl(n), \quad (A, B) \mapsto AB$$
is a smooth map.