Differential Geometry (104.358)
Exercise sheet for 15.3.2018

1. Consider the following maps:

$$a: \mathbb{R} \to \mathbb{R}^2 \qquad t \mapsto \begin{pmatrix} t \\ t^2+1 \end{pmatrix} \qquad\qquad b: \mathbb{R} \to \mathbb{R}^2 \qquad t \mapsto \begin{pmatrix} t^2 \\ t^4+1 \end{pmatrix}$$

$$c: \mathbb{R} \to \mathbb{R}^2 \qquad t \mapsto \begin{pmatrix} t^3 \\ t^6+1 \end{pmatrix} \qquad\qquad d: \mathbb{R} \to \mathbb{R}^2 \qquad t \mapsto \begin{pmatrix} sign(t)\sqrt{|t|} \\ |t|+1 \end{pmatrix}$$

$$e: \mathbb{R} \to \mathbb{R}^2 \qquad t \mapsto \begin{pmatrix} \sinh(t) \\ \cosh^2(t) \end{pmatrix} \qquad\qquad f: \mathbb{R} \to \mathbb{R}^2 \qquad t \mapsto \begin{pmatrix} \sin(t) \\ \sin^2(t)+1 \end{pmatrix}$$

For d: $sign$ is the sign function: $sign(t) = 1$ for $t > 0$, $sign(0) = 0$, $sign(t) = -1$ for $t < 0$. Thus, $t = sign(t)|t|$.

We consider $\mathbb{R}^2$ with the standard euclidean norm

$$\|\cdot\| : \mathbb{R}^2 \to \mathbb{R}, \qquad (x, y) \mapsto \|(x, y)\| = \sqrt{x^2 + y^2}.$$

Investigate:

(a) differentiability

(b) regularity: where is $\|X'\| \neq 0$, $X \in \{a, b, c, d, e, f\}$?

(c) injectivity

of the above maps.

For which pairs of the above maps $X_1, X_2 \in \{a, b, c, d, e, f\}$ is there a diffeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ such that $X_1 = X_2 \circ \psi$? Which maps have the same image?

2. Consider the catenary:

$$X: \mathbb{R} \to \mathbb{R}^2, \quad t \mapsto \begin{pmatrix} \cosh(t) \\ t \end{pmatrix}.$$

Calculate the first derivative vector X' (tangent vector), its norm $\|X'\|$ and the normalised derivative vector $T = \frac{X'}{\|X'\|}$. What are the asymptotic directions

$$\lim_{t \to \pm\infty} T(t) \quad ?$$

Through what angle does T rotate in the interval $(-\infty, 0)$ and $(-\infty, \infty)$?

Calculate also the curvature $\kappa(t) := \frac{\det(X'(t) X''(t))}{\|X'(t)\|^3}$.

3. Consider

(a) Neile's semicubical parabola, $\quad p: \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} t^3 \\ t^2 \end{pmatrix}$.

(b) The graph of the absolute value function, $b:\ \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} t \\ |t| \end{pmatrix}$.

(c) The straight line, $g:\ \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} t^3 \\ 0 \end{pmatrix}$.

Investigate the differentiability and regularity (is $\|X'(0)\| \neq 0$ for $X \in \{p, b, g\}$) of the three maps at $t = 0$. Now remove the "problematic" point $t = 0$ from the domain $\mathbb{R} = \mathbb{R}_- \cup \{0\} \cup \mathbb{R}_+$, to obtain in each case two differentiable, regular maps $p_\pm = p|_{\mathbb{R}_\pm}$, $b_\pm = b|_{\mathbb{R}_\pm}$, $g_\pm = |_{\mathbb{R}_\pm}$ with domain the positive real numbers $\mathbb{R}_+$ or the negative real numbers $\mathbb{R}_-$. Calculate the normalised tangent vectors $T = \frac{X'}{\|X'\|}$ of these maps and their limits as $t \to 0$, i.e.,

$$\lim_{\epsilon \to 0} \frac{X'_\pm(\pm\epsilon)}{\|X'_\pm(\pm\epsilon)\|}, \qquad X \in \{p, b, g\}.$$

What conclusions can be made by looking at the images of these maps at $t = 0$?