Differential Geometry (104.358)
Exercise sheet for 21.3.2019

4. Consider again the following curves:

   (a) Neile's semicubical parabola, $p:\ \mathbb{R} \to \mathcal{E}^3,\ t \mapsto O + e_1 t^3 + e_2 t^2.$

   (b) The graph of the absolute value function, $b:\ \mathbb{R} \to \mathcal{E}^3,\ t \mapsto O + e_1 t + e_2 |t|.$

   (c) The straight line, $g:\ \mathbb{R} \to \mathcal{E}^3,\ t \mapsto O + e_1 t^3.$

   For which of the three maps is there a diffeomorphism $\psi : \mathbb{R} \to \mathbb{R}$ so that $X \circ \psi : \mathbb{R} \to \mathcal{E}^3$, $X \in \{p, b, g\}$, is regular?

   For which of the three maps is there a bijective, differentiable map $\phi : \mathbb{R} \to \mathbb{R}$ so that $\tilde{X} = X \circ (\phi^{-1}) : \mathbb{R} \to \mathcal{E}^3$, $X \in \{p, b, g\}$, is regular? (And for which is there none?)

5. Consider the tractrix

$$X : \mathbb{R} \to \mathcal{E}^3, \quad X(t) = O + (t - \tanh t) e_1 + \frac{1}{\cosh t} e_2.$$

   Is the tractrix a regular curve?

   Where X is regular, compute the tangent line $\mathcal{T}(t) = X(t) + [X'(t)]$ and the point of intersection $P(t)$ of the tangent line with the line $\{O + \lambda e_1 : \lambda \in \mathbb{R}\}$.

   What is the length of the line segment between $X(t)$ and $P(t)$?

6. Prove: A map

$$\begin{aligned} X :\ \mathbb{R} \supseteq \mathcal{I} &\to \mathcal{E}^n, \\ t &\mapsto X(t), \end{aligned}$$

   with $X' \neq 0$ parametrises a line or a line segment if $X'(t)$ and $X''(t)$ are linearly dependent.

   In this case, what sort of reparametrisations $\tilde{X}(\tilde{t})$ of $X(t)$ satisfy $\tilde{X}'' = 0$?

   Show that if all of the tangent lines $\mathcal{T}(t)$ of a curve $X : I \to \mathcal{E}^n$ pass through a fixed point $P \in \mathcal{E}^n$, then the curve must be a straight line.

7. Convince yourself that the arc length

$$s(t) = \int_{t_0}^{t} |X'(\tilde{t})| \,\mathrm{d}\,\tilde{t}$$

   of a curve $t \mapsto X(t)$ is invariant under reparametrisation.

8. Consider the curve $X(t) = O + e_1 x(t) + e_2 y(t) + e_3 z(t)$ defined implicitly by the equations

$$\left(\frac{x(t)}{a}\right)^2 + \left(\frac{y(t)}{b}\right)^2 + \left(\frac{z(t)}{c}\right)^2 = 1, \qquad a\sqrt{b^2 - c^2}\ z(t) = c\sqrt{a^2 - b^2}\ x(t),$$

   with $a > b > c > 0$.

   (a) Which two surfaces intersect to give the curve?

   (b) Find a parametrisation of the curve.

   (c) Calculate the arc length and an arc length parametrisation of the curve.

   (d) What does the curve look like?