Differential Geometry (104.358)
Exercise sheet for 02.5.2019

24. Investigate how the first and second fundamental forms of a surface change under reparametrisation and Euclidean motions.

25. Assume a surface

$$X : \mathbb{R}^2 \supseteq U \to \mathcal{E}^3,$$
$$(u, v) \mapsto X(u, v) = O + e_1 x + e_2 y + e_3 z,$$

is implicitly defined through an equation $F(x, y, z) = 0$ with a smooth map $F : \mathbb{R}^3 \to \mathbb{R}$ with $\operatorname{grad} F \neq 0$ whenever $F(x, y, z) = 0$. We can equivalently define the surface through the equation $F \circ (X - O) = 0$. Show that the Gauss map is given by

$$N = \pm \frac{(\operatorname{grad} F) \circ (X - O)}{\|(\operatorname{grad} F) \circ (X - O)\|}$$

(grad denotes the gradient, i.e., the vector of partial derivatives).

26. Show that all the points on the sphere with radius $r > 0$ are umbilic points and calculate its Gauss and mean curvature.

Hint: Don't use an explicit parametrisation of the sphere.

27. Let

$$X : \mathbb{R}^2 \supseteq U \to \mathcal{E}^3,$$
$$(u, v) \mapsto O + e_1 u + e_2 v + e_3 z(u, v),$$

where $z : U \to \mathbb{R}$ is smooth.

Calculate the Gauss map and the shape operator of X.

Now let (u_0, v_0) be a stationary point of the function z. What does the shape operator and the Gauss curvature look like at this point?

Question to be handed in (written neatly or typed) on 02/05/19:

Consider the (left handed) helicoid

$$X(u, v) = O + e_1 \sinh u \sin v - e_2 \sinh u \cos v + e_3 v,$$

and the catenoid

$$\tilde{X}(u, v) = O + e_1 \cosh u \cos v + e_2 \cosh u \sin v + e_3 u.$$

a) Compute the first fundamental forms of X and $\tilde{X}$.

b) Compute the Gauss maps of X and $\tilde{X}$.

c) X and $\tilde{X}$ are *minimal surfaces*, that is, that their mean curvatures are zero. Prove this for the (left handed) helicoid X.

d) Find a 1-parameter family of minimal surfaces $X^\theta : \mathbb{R}^2 \to \mathcal{E}^3$ depending smoothly on a parameter $\theta \in [-\pi, \pi]$ such that

- $X^0 = X$ is the (left handed) helicoid,
- $X^{\pi/2} = \tilde{X}$ is the catenoid, and
- $\mathrm{I}^\theta = \mathrm{I}^0$ for all θ.

[Hint: write $X^\theta = X a(\theta) + \tilde{X} b(\theta)$ for some functions a and b of θ.]