Differential Geometry (104.358)
Exercise sheet for 04.4.2019

14. Let $X : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^3$ be a plane curve and N a unit normal field that is tangent to this plane. Show that N is parallel for the normal connection. What do the other parallel normal fields along X look like?

15. Let $X : \mathbb{R} \supseteq \mathcal{I} \to \mathcal{E}^3$ be a curve in $\mathcal{E}^3$. Prove:

    (a) Any two parallel frames of X differ by a constant rotation in the normal plane.

    (b) Let N be a parallel normal field along X. Then a normal field $\tilde{N}$ is parallel if and only if it makes a constant angle with N.

16. Suppose that $X : I \to \mathcal{E}^3$ is a Frenet curve. Show that the curvature and torsion of X are given by
$$\kappa = \frac{|X' \times X''|}{|X'|^3} \quad \text{and} \quad \tau = \frac{\det(X', X", X''')}{|X' \times X''|^2}.$$
[Hint: for the torsion one can use the scalar triple product: $\langle a, (b \times c)\rangle = \det(a, b, c)$.]

17. Suppose that $X : I \to \mathcal{E}^3$ is a Frenet curve lying on a sphere of radius $r > 0$ with centre $O \in \mathcal{E}^3$ and assume that the torsion τ of X is always positive. Show that
$$\left(\frac{1}{\kappa}\right)^2 + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)^2 \equiv r^2. \tag{1}$$
[Hint: Show that $X - O = -N\frac{1}{\kappa} - B(\frac{1}{\kappa})'\frac{1}{\tau}$ by differentiating the condition $|X - O|^2 = r^2$ three times. Here (T, N, B) is the Frenet frame.]

    Is the converse true: if (1) is satisfied for some $r > 0$, does X have to lie on a sphere?

18. Express the curvature and torsion of a Frenet curve in terms of κ_n and κ_g of a parallel frame and vice versa.