Differential Geometry (104.358)
Exercise sheet for 06.6.2019

40. Prove Euler's theorem:

    The normal curvatures κ_n at a point of a surface satisfy

    $$\kappa_n = \kappa^+ \cos^2\theta + \kappa^- \sin^2\theta,$$

    where $\kappa^\pm$ are the principal curvatures and θ is the angle between the tangent direction of $\kappa_n(\theta)$ and the curvature direction of κ^+.

    Hint: fix $(u,v) \in M$ and use a basis (e_1, e_2) of $\mathbb{R}^2$ that is orthonormal for $\mathrm{I}|_{(u,v)}$ and diagonalises $\mathrm{I\!I}|_{(u,v)}$.

41. Fix a point $X(u,v)$ on a parametrised surface and assume that this is not a flat point. Prove that an asymptotic line can pass through $X(u,v)$ in two, one or no directions depending on the sign of the Gauss curvature $K(u,v)$.

42. Let X_1 and X_2 be two surfaces that intersect along a curve C. Suppose that the Gauss maps of the two surfaces are linearly independent along C.

    Show that C is a pre-geodesic line of both X_1 and X_2 if and only if C is a line segment.

43. Let X be a surface and suppose that along a curve $t \mapsto C(t) = X(u(t), v(t))$ the surface is tangent to a fixed plane, i.e., the tangent planes of X along C are all the same.

    Show that the Gauss Curvature vanishes along the curve C.