Differential Geometry (104.358)
Exercise sheet for 09.5.2019

28. For $c \in \mathcal{E}^3$ und $\rho \in \mathbb{R}$, let $D_{c,\rho}$ be the function

$$\begin{aligned} D_{c,\rho} : \mathcal{E}^3 &\to \mathbb{R}, \\ Y &\mapsto |Y - c|^2 - \rho^2. \end{aligned}$$

Then the set of points Y satisfying $D_{c,\rho}(Y) = 0$ is a sphere with centre c and radius ρ. Let $X : M \supseteq U \to \mathcal{E}^3$ be a surface.

(a) For a point $(u_0, v_0) \in M$, such that $X(u_0, v_0)$ is not an umbilic point, show that

$$(D_{c,\rho} \circ X)(u_0, v_0) = 0, \qquad \begin{pmatrix} (D_{c,\rho} \circ X)_u \\ (D_{c,\rho} \circ X)_v \end{pmatrix} (u_0, v_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and}$$

$$\begin{pmatrix} (D_{c,\rho} \circ X)_{uu} & (D_{c,\rho} \circ X)_{uv} \\ (D_{c,\rho} \circ X)_{vu} & (D_{c,\rho} \circ X)_{vv} \end{pmatrix} (u_0, v_0) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if and only if $d_{(u_0,v_0)}X\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is a curvature direction,

$$c = X(u_0, v_0) + \frac{1}{\kappa(u_0, v_0)} N(u_0, v_0) \quad \text{and} \quad \rho^2 = \frac{1}{\kappa^2(u_0, v_0)},$$

where N is the Gauss map of X and $\kappa(u_0, v_0)$ is the principal curvature corresponding to the curvature direction $d_{(u_0,v_0)}X\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$.

(b) Show that c and ρ can be chosen so that

$$(D_{c,\rho} \circ X)(u_0, v_0) = 0, \qquad \begin{pmatrix} (D_{c,\rho} \circ X)_u \\ (D_{c,\rho} \circ X)_v \end{pmatrix} (u_0, v_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and}$$

$$\begin{pmatrix} (D_{c,\rho} \circ X)_{uu} & (D_{c,\rho} \circ X)_{uv} \\ (D_{c,\rho} \circ X)_{vu} & (D_{c,\rho} \circ X)_{vv} \end{pmatrix} (u_0, v_0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

if and only if $X(u_0, v_0)$ is an umbilic point.

Note that spheres satisfying the properties in (a) and (b) are called *curvature spheres* of X at (u_0, v_0).

29. Compute the Christoffel symbols of a conformally parametrised surface.

30. Compute the Christoffel symbols of the plane parametrised by polar coordinates

$$X(u, v) = O + e_1 u \cos v + e_2 u \sin v.$$

31. Let X be a surface whose image lies on a sphere with radius $r > 0$. Show that the curvature tensor R satisfies

$$RY = \frac{1}{r^2} Y \times X_u \times X_v,$$

for any tangent field $Y : M \to \mathbb{R}^3$.