Differential Geometry (104.358)
Exercise sheet for 11.4.2019

19. Show that the ellipsoid with two points removed

$$E = \left\{ O + e_1 x + e_2 y + e_3 z \ \in \mathcal{E}^3 \mid \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1,\ |z| < c \right\}, \qquad a > b > c,$$

is a surface by finding a regular parametrisation.

20. Show that the torus

$$T^2 := \left\{ O + e_1 x + e_2 y + e_3 z \ \in \mathcal{E}^3 \mid \left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = r^2 \right\}$$

with $0 < r < R$ is a surface.

<u>Hint:</u> How is the torus made up of circles? Use parametrisations of circles to find a parametrisation of the torus.

21. Show that the circular cone with its apex removed,

$$\{O + e_1 x + e_2 y + e_3 z \mid x^2 + y^2 = z^2,\ z > 0\},$$

admits an isometric parametrisation.

22. Consider the sphere $S^2 \subset \mathbb{R}^3$ with radius 1 and centred at the origin. Let $n = (0,0,1) \in S^2$ be the north pole of the sphere. For every point $(u,v,0)$ in the e_1e_2-plane draw a line through $(u,v,0)$ and n. This line cuts the sphere in the north pole and another point $\Pi(u,v)$ and thus defines a map

$$\Pi : \mathbb{R}^2 \to S^2 \subset \mathbb{R}^3,$$
$$(u,v) \mapsto \Pi(u,v).$$

Calculate the map Π. What is the image of Π? What is $\lim_{u\to\infty} \Pi(u,v)$?

After a choice of origin $O \in \mathcal{E}^3$, define a map X to $\mathcal{E}^3$,

$$X : \mathbb{R}^2 \to \mathcal{E}^3,$$
$$(u,v) \mapsto O + \Pi(u,v).$$

Convince yourself that X is a regular parametrisation of the unit sphere centred at O without the north pole. Is the parametrisation conformal?

What is the image of a line in $\mathbb{R}^2$ under X? What is the image of a circle in $\mathbb{R}^2$ with centre at the origin under X?

Note: the inverse function Π^{-1} is called stereographic projection.

23. Let $r > 0$ and define a parametrisation of the Möbius strip by

$$X : \mathbb{R}^2 \to \mathcal{E}^3,$$
$$(u,v) \mapsto O + (e_1 \cos 2u + e_2 \sin 2u)(r + v\cos u) + e_3 v \sin u.$$

Calculate the Gauss map N of X along $v = 0$, i.e., calculate $N(u,0)$.

Show that $X(u+\pi,0) = X(u,0)$ but $N(u+\pi,0) = -N(u,0)$.