Differential Geometry (104.358)
Exercise sheet for 14.03.2019

1. Consider the following maps:

$$a:\ \mathbb{R}\to\mathbb{R}^2 \qquad t\mapsto \begin{pmatrix} t \\ t^2+1 \end{pmatrix} \qquad\qquad b:\ \mathbb{R}\to\mathbb{R}^2 \qquad t\mapsto \begin{pmatrix} t^2 \\ t^4+1 \end{pmatrix}$$

$$c:\ \mathbb{R}\to\mathbb{R}^2 \qquad t\mapsto \begin{pmatrix} t^3 \\ t^6+1 \end{pmatrix} \qquad\qquad d:\ \mathbb{R}\to\mathbb{R}^2 \qquad t\mapsto \begin{pmatrix} sign(t)\sqrt{|t|} \\ |t|+1 \end{pmatrix}$$

$$e:\ \mathbb{R}\to\mathbb{R}^2 \qquad t\mapsto \begin{pmatrix} \sinh(t) \\ \cosh^2(t) \end{pmatrix} \qquad\qquad f:\ \mathbb{R}\to\mathbb{R}^2 \qquad t\mapsto \begin{pmatrix} \sin(t) \\ \sin^2(t)+1 \end{pmatrix}$$

For d: $sign$ is the sign function: $sign(t)=1$ for $t>0$, $sign(0)=0$, $sign(t)=-1$ for $t<0$. Thus, $t=sign(t)|t|$.

We consider $\mathbb{R}^2$ with the standard euclidean norm

$$\|\cdot\|:\mathbb{R}^2\to\mathbb{R},\qquad (x,y)\mapsto\|(x,y)\|=\sqrt{x^2+y^2}.$$

Investigate:

(a) differentiability

(b) regularity: where is $\|X'\|\neq 0$, $X\in\{a,b,c,d,e,f\}$?

(c) injectivity

of the above maps.

For which pairs of the above maps $X_1,X_2\in\{a,b,c,d,e,f\}$ is there a diffeomorphism $\psi:\mathbb{R}\to\mathbb{R}$ such that $X_1=X_2\circ\psi$? Which maps have the same image?

2. Consider the catenary:

$$X:\ \mathbb{R}\to\mathbb{R}^2,\quad t\mapsto\begin{pmatrix}\cosh(t)\\ t\end{pmatrix}.$$

Calculate the first derivative vector X' (tangent vector), its norm $\|X'\|$ and the normalised derivative vector $T=\frac{X'}{\|X'\|}$. What are the asymptotic directions

$$\lim_{t\to\pm\infty}T(t)\qquad ?$$

Through what angle does T rotate in the interval $(-\infty,0)$ and $(-\infty,\infty)$?

Calculate also the curvature $\kappa(t):=\frac{\det(X'(t)X''(t))}{\|X'(t)\|^3}$.

3. Consider

(a) Neile's semicubical parabola, $\quad p:\ \mathbb{R}\to\mathbb{R}^2,\ t\mapsto\begin{pmatrix}t^3\\ t^2\end{pmatrix}.$

(b) The graph of the absolute value function, $b:\ \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} t \\ |t| \end{pmatrix}.$

(c) The straight line, $g:\ \mathbb{R} \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} t^3 \\ 0 \end{pmatrix}.$

Investigate the differentiability and regularity (is $\|X'(0)\| \neq 0$ for $X \in \{p, b, g\}$) of the three maps at $t = 0$. Now remove the "problematic" point $t = 0$ from the domain $\mathbb{R} = \mathbb{R}_- \cup \{0\} \cup \mathbb{R}_+$, to obtain in each case two differentiable, regular maps $p_\pm = p|_{\mathbb{R}_\pm}$, $b_\pm = b|_{\mathbb{R}_\pm}$, $g_\pm = |_{\mathbb{R}_\pm}$ with domain the positive real numbers $\mathbb{R}_+$ or the negative real numbers $\mathbb{R}_-$. Calculate the normalised tangent vectors $T = \frac{X'}{\|X'\|}$ of these maps and their limits as $t \to 0$, i.e.,

$$\lim_{\epsilon \to 0} \frac{X'_\pm(\pm\epsilon)}{\|X'_\pm(\pm\epsilon)\|}, \qquad X \in \{p, b, g\}.$$

What conclusions can be made by looking at the images of these maps at $t = 0$?