Differential Geometry (104.358)
Exercise sheet for 16.5.2019

32. Let Σ denote the matrix representation of the shape operator. Prove that the partial covariant derivatives, $\nabla_{\frac{\partial}{\partial u}} S$ and $\nabla_{\frac{\partial}{\partial v}} S$ have matrix representations

$$\Sigma_u + [\Gamma_1, \Sigma] \quad \text{and} \quad \Sigma_v + [\Gamma_2, \Sigma],$$

where $[.,.]$ denotes the usual commutator of matrices. Derive an expression of the Codazzi equation in terms of these.

33. Prove that for a conformally parametrised surface the Gauss equation reads

$$K = -\frac{1}{2E}\Delta \ln E.$$

34. Suppose that $X : M \to \mathcal{E}^3$ is a surface with

$$\mathrm{I} = du^2 + 2\cos\theta du dv + dv^2 \quad \text{and} \quad \mathrm{I\!I} = 2\sin\theta du dv$$

for some smooth function $\theta : M \to \mathbb{R}$.

Determine the Gauss curvature of X and then use the Gauss equation to derive a second order differential equation for θ.

35. Let $F = (X_u, X_v, N)$ be an adpated frame for the surface $X : M \to \mathbb{R}^3$. Show that

$$\begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix} F^{-1}F_u \qquad \text{and} \qquad \begin{pmatrix} E & F & 0 \\ F & G & 0 \\ 0 & 0 & 1 \end{pmatrix} F^{-1}F_v$$

are skew-symmetric matrices if and only if

$$I = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

is constant.