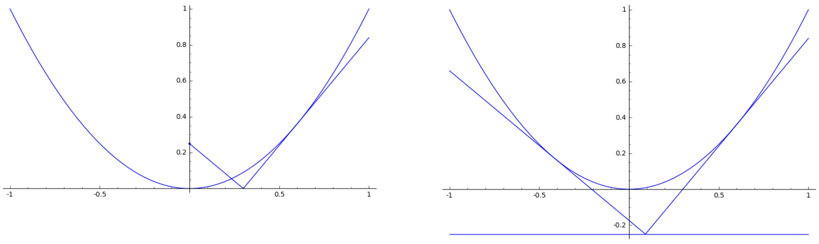
TU WIEN   INSTITUT FÜR DISKRETE MATHEMATIK UND GEOMETRIE

Differential Geometry

Spring term 2020

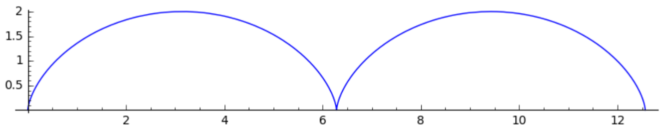
Exercise 1

1. (a) Show that every tangent to the parabola $y = x^2$ is perpendicular to the line through the focus $(0, \frac{1}{4})$ and the intersection point of the tangent with the x-axis.
   (b) Show that if two tangents meet on the directrix $y = -\frac{1}{4}$, then they are perpendicular.



Recall the parametric equation of the cycloid:

$$\gamma(t) = (t - \sin t, 1 - \cos t).$$



2. Compute the length of one arc of the cycloid.

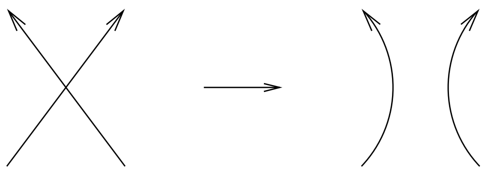
3. Compute the curvature of the cycloid at each point. How does the curvature behave near the singular points?

4. Show that the perimeter of the ellipse with half-axes $a > b$ is equal to

$$4a \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 t}\, dt = 4a \int_0^1 \frac{\sqrt{1 - k^2 x^2}}{\sqrt{1 - x^2}}\, dx,$$

   where $k = \sqrt{1 - \frac{b^2}{a^2}}$ is the excentricity of the ellipse. (This integral is called the complete elliptic integral of the second kind.)

5.* Show that the turning number T of a generic immersed closed curve can be computed as follows. Orient the curve arbitrarily. Then replace the neighborhood of each intersection point by two disjoing arcs as shown in the figure.



The curve decomposes into a collection of simple oriented curves. Let I_+ be the number of counterclockwise oriented curves, and I_- the number of clockwise oriented ones. Then $T = I_+ - I_-$.