Differential Geometry

Spring term 2020

Exercise 2

1. Compute the Frenet-Serret frame, the curvature, and the torsion of the helix
$$\gamma(t) = (a\cos t, a\sin t, bt).$$

2. Let γ be a unit-speed curve in $\mathbb{R}^3$. With the help of the Frenet-Serret formulas find the components of the vector $\dddot{\gamma}$ in the Frenet-Serret frame. Derive from this the formula
$$\tau = \frac{\det(\dot{\gamma}, \ddot{\gamma}, \dddot{\gamma})}{\kappa^2}.$$

3. Let γ be a unit-speed curve in $\mathbb{R}^n$. For any $\varepsilon \in \mathbb{R}$ consider the curve
$$\gamma^\varepsilon(t) = \gamma(t) + \varepsilon\dot{\gamma}(t).$$
   (a) Show that the curve γ^ε is regular.
   (b) Show that γ^ε is at least as long as γ:
$$L(\gamma^\varepsilon) \geq L(\gamma).$$

4. Let γ be a space curve contained in the unit sphere $\{p \in \mathbb{R}^3 \mid \|p\| = 1\}$. Show that the curvature of γ is at least 1 at every point. (*Hint*: take the arc-length parametrization and differentiate the inner product $\langle \gamma, \dot{\gamma} \rangle$.)

5.** The pair of curves $\gamma, \gamma^\varepsilon$ from Exercise 3 can be viewed as the tracks of the rear and front wheel of a bicycle (the bicycle frame is always tangent to the trajectory of the rear wheel, while the front wheel can steer).

   Given the tracks of the rear and front wheel, can you tell which way the bicycle has traveled? (Sherlock Holmes gives a confusing explanation in "The Adventure of the Priory School".)

   For fascinating geometry of bicycle tracks see [2, 1].

References

[1] Robert Foote, Mark Levi, and Serge Tabachnikov. Tractrices, bicycle tire tracks, hatchet planimeters, and a 100-year-old conjecture. *Amer. Math. Monthly*, 120(3):199–216, 2013.

[2] Mark Levi and Serge Tabachnikov. On bicycle tire tracks geometry, hatchet planimeter, Menzin's conjecture, and oscillation of unicycle tracks. *Experiment. Math.*, 18(2):173–186, 2009.