Problem 1. Let $\alpha : I = [t_0, t_1] \to \mathbb{R}^3$, $t \mapsto \alpha(t)$ be a regular curve and $\beta : J = [0, L_\alpha] \to \mathbb{R}^3$, $s \mapsto \beta(s)$ be a reparametrization of α by arc length s,

$$s(t) = \int_{t_0}^{t} \|\alpha'(t)\| dt, \qquad t \in [t_0, t_1].$$

- Show that the curvature of α at $t \in I$ can be computed as

$$\kappa(t) = \frac{\|\alpha'(t) \times \alpha''(t)\|}{\|\alpha'(t)\|^3}$$

- Show that the signed curvature of a planar curve $\alpha(t) = \big(x(t), y(t)\big)$ is given as

$$\kappa(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{\big((x'(t))^2 + (y'(t))^2\big)^{3/2}}$$

Hints: (i) By definition $\kappa(t) = \kappa_\beta(s(t)) = \|\ddot{\beta}(s(t))\|$. To expand this expression you will need formulas for $s'(t)$ and $s''(t)$. (ii) At some point it will be helpful to recall the formula for the area of a parallelogram.

Problem 2. Given the parametrized curve

$$\gamma(s) = \big(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}\big), \qquad s \in \mathbb{R},$$

where $a > 0$, $b > 0$, and $c^2 = a^2 + b^2$.

- Show that s is the arc length parameter.
- Compute the curvature of γ.
- Show that the lines spanned by the principal normal $\mathbf{e_2}(s)$ and passing through $\gamma(s)$ meet the z-axis under a constant angle.
- Show that the tangent lines of γ form a constant angle with the z-axis.

Problem 3. The catenary curve is defined as $\gamma(t) = (t, \cosh t), \in \mathbb{R}$

- Show that the signed curvature of γ is

$$\kappa(t) = \frac{1}{\cosh^2 t}.$$

- Show that the evolute of γ is

$$\gamma^*(t) = (t - \sinh t \cosh t, 2 \cosh t)$$

Problem 4. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

- Show that $\gamma : \mathbb{R} \to \mathbb{R}^2, t \mapsto \begin{pmatrix} t \\ f(t) \end{pmatrix}$ defines a regular curve.
- Derive a closed formula for the curvature κ of γ.
- Let $f = \frac{1}{2}at^2$, $a \neq 0$. Compute the curvature of γ at $t = 0$.

Problem 5. Let $\mathbf{x} : \mathbb{R}^2 \to \mathbb{R}^3, (u, v) \mapsto (u, v, u^2 + 2v^2)$

- Show that $\mathbf{x}$ defines a regular surface.
- Compute the normal curvature κ_n in direction $\mathbf{x}_u + \mathbf{x}_v$ at $(0, 0)$.

Problems due October 19, 2017.

# of completed problems	grade
$< 50\%$	5
$50 - 59\%$	4
$60 - 74\%$	3
$75 - 84\%$	2
$> 84\%$	1