Stochastic Analysis, WS 2019
Exercise sheet 1

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Let $(\Omega, \mathcal{A}, P)$ be a probability space.

Exercise 1) Let $Y \sim \mathcal{N}(0,1)$, $M : \Omega \to \{-1, 1\}$ with $P(M = 1) = P(M = -1)$ and $Z := YM$. Assume that Y, M are independent.

Show that Y, Z are uncorrelated standard normal random variables, i.e. $Z \sim \mathcal{N}(0,1)$ and

$$\mathrm{Cov}(Y, Z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

but (Y, Z) is not a normal random variable.

Exercise 2) Let X and Y be random variables with finite first moment defined on a probability space $(\Omega, \mathcal{A}, P)$.

(a) Show that if X and Y are independent, then their covariance is zero.

(b) Find an example when X and Y are uncorrelated, but not independent. (Please don't use example from the Exercise 1.)

Exercise 3) (Moments of the one-dimensional centered normal distribution). Let $Y \sim \mathcal{N}(0, \sigma^2)$ with $\sigma^2 \geq 0$. Show for all $n \in \mathcal{N}$ that $E[Y^{2n+1}] = 0$ and

$$E[Y^{2n}] = \frac{(2n)!}{n!} \left(\frac{\sigma^2}{2} \right)^n.$$

Hint: Combine $e^{|Y|} \leq e^Y + e^{-Y}$, Exercises 2.16 and 2.18(a) to show the existence of the moments. The power series representation of the exponential function,

$$\mathrm{E}[e^{tY}] = \sum_{n=0}^{\infty} \frac{t^n \mathrm{E}[X^n]}{n!}, t \in \mathcal{R} \tag{1}$$

and the identity theorem for power series give the moments. Alternatively, partial integration can be used.

Exercise 4) Let $Y \sim \mathcal{N}(\mu, C)$ where $\mu \in \mathbb{R}^n$ and $C \in \mathbb{R}^{n \times n}$. Show that

$$\mathrm{E}\left[e^{\langle t, Y \rangle} \right] = \exp(\langle t, \mu \rangle + \frac{1}{2} \langle t, Ct \rangle), \quad t \in \mathbb{R}^n.$$

Hint: Prove the one-dimensional case by explicit calculation and completion of the square.