1. Problem Set for the Course Mathematical Finance 2: Continuous-Time Models

March 15, 2012

1. Problem: *Martingale representation*
Let $W = \{W_t\}_{t\ge 0}$ with $W_t = (W_t^{(1)}, \dots, W_t^{(n)})^\top$ be a standard n-dimensional Brownian motion and $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$. In every one of the cases

(a) $M_t = \sigma W_t$ for $t \ge 0$,

(b) $M_t = \|W_t\|_2^2 - nt$ for $t \ge 0$,

(c) $M_t = \exp(\sigma W_t - \frac{1}{2}\|\sigma\|_2^2 t)$ for $t \ge 0$,

find a progessively measurable process $f\colon [0,\infty)\times\Omega \to \mathbb{R}^n$ with $\mathbb{E}[\int_0^t \|f(s)\|_2^2\,ds] < \infty$ for every $t \ge 0$ such that

$$M_t \overset{\text{a.s.}}{=} \mathbb{E}[M_0] + \int_0^t f(s)\,dW_s, \qquad t \ge 0.$$

Hint: Apply the n-dimensional Itō formula.

Remark: (b) is a compensated squared Bessel process of dimension n, (c) is geometric Brownian motion.

2. Problem: *Uniqueness of the solution of a stochastic differential equation*
Let $W = \{W_t\}_{t\ge 0}$ with $W_t = (W_t^{(1)}, \dots, W_t^{(n)})^\top$ be a standard n-dimensional Brownian motion, $\mu \in \mathbb{R}$ and $\sigma = (\sigma_1, \dots, \sigma_n) \in \mathbb{R}^n$. Give a direct proof that

$$S_t = S_0 \exp\Big(\sigma W_t + \Big(\mu - \frac{\|\sigma\|_2^2}{2}\Big)t\Big), \qquad t \ge 0,$$

is (up to indistinguishability) the unique strong solution of the stochastic differential equation

$$dS_t = \mu S_t\,dt + S_t\sigma\,dW_t, \quad t \ge 0,$$

with deterministic initial condition $S_0 > 0$.

Hint: For uniqueness, let $\tilde S = \{\tilde S_t\}_{t\ge 0}$ denote another solution of the SDE and apply the two-dimensional Itō formula to the process defined by $X_t = \tilde S_t / S_t$ for $t \ge 0$.