2. Problem Set for the Course Mathematical Finance 2: Continuous-Time Models

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3. Problem: *Representation as stochastic integral*
Let $W = \{W_t\}_{t\in[0,T]}$ with $W_t = (W_t^{(1)},\dots,W_t^{(n)})^\top$ denote a standard n-dimensional Brownian motion up to time $T>0$ and let $\sigma = (\sigma_1,\dots,\sigma_n)\in\mathbb{R}^n$. In every one of the cases

(a) $F = \int_0^T \sigma W_t\,dt$,
(b) $F = (\sigma W_T)^3$,
(c) $F = \sin(\sigma W_T)$,

find a progessively measurable process $f\colon [0,T]\times\Omega\to\mathbb{R}^n$ with $\mathbb{E}[\int_0^T \|f(t)\|_2^2\,dt]<\infty$ such that

$$F \overset{\text{a.s.}}{=} \mathbb{E}[F] + \int_0^T f(t)\,dW_t.$$

Hint: Apply Itō's formula to (a) $t\sigma W_t$ and (c) $e^{at}\sin(\sigma W_t)$ with suitable $a\in\mathbb{R}$.

4. Problem: *Harmonic and holomorphic functions of Brownian motion*

(a) Let $W = \{W_t\}_{t\ge 0}$ denote a standard n-dimensional Brownian motion and let $f\colon\mathbb{R}^n\to\mathbb{R}$ be a harmonic function. Prove that $Z_t = f(W_t)$ for $t\ge 0$ defines a continuous local martingale.
(b) Consider the two-dimensional standard Brownian motion $W = \{W_t\}_{t\ge 0}$ with $W_t = W_t^{(1)} + iW_t^{(2)}$ as a $\mathbb{C}$-valued one. Let $f\colon\mathbb{C}\to\mathbb{C}$ be a holomorphic function. Prove that $Z_t = f(W_t)$ for $t\ge 0$ defines a $\mathbb{C}$-valued continuous local martingale.

Hints: (a) Use Itō's formula. (b) Use the Cauchy–Riemann equations and part (a).

5. Problem: *A local martingale with continuous paths that is not a martingale*
Let $W = \{W_t\}_{t\ge 0}$ denote a Brownian motion in dimension $n\ge 3$ with starting point $W_0 = x\in\mathbb{R}^n\setminus\{0\}$. For $r>0$ define $\tau_r = \inf\{t\ge 0 \mid \|W_t\|_2\le r\}$. Show the following:

(a) $M_t^{(r)} := \|W_{t\wedge\tau_r}\|_2^{2-n}$ for $t\ge 0$ is a bounded martingle.
(b) $\mathbb{P}(\tau_r<\infty) = (r/\|x\|_2)^{n-2}$ for all $r\in(0,\|x\|_2]$.
(c) $\mathbb{P}(W_t\ne 0 \text{ for all } t\ge 0) = 1$.
(d) $M = \{M_t\}_{t\ge 0}$ with $M_t := \|W_t\|_2^{2-n}$ is a local martingale and can be represented as

$$M_t = \|x\|_2^{2-n}\exp\Bigl((2-n)\int_0^t \frac{W_s\,dW_s}{\|W_s\|_2^2} - \frac{(2-n)^2}{2}\int_0^t \frac{ds}{\|W_s\|_2^2}\Bigr), \qquad t\ge 0.$$

(e) M is a supermartingale.
(f) $\mathbb{E}[M_t]\to 0$ as $t\to\infty$, hence M is not a martingale.
(g) $M_\infty := \lim_{t\to\infty} M_t$ exists almost surely and $M_\infty \overset{\text{a.s.}}{=} 0$.
(h) $\{M_t\}_{t\ge 0}$ is uniformly integrable.

Hints: (a) Show that $f(y) := \|y\|_2^{2-n}$ for $y \in \mathbb{R}^n \setminus \{0\}$ is harmonic and use Itō's formula. (b) Define $\tau_R = \inf\{t \geq 0 \mid \|W_t\|_2 \geq R\}$ for $R > \|x\|_2$ and apply Doob's optional sampling theorem to $\tau_r \wedge \tau_R$. Then consider $R \to \infty$. (d) For the representation apply Itō's formula to $\log \|W_t\|_2$. (e) Use the conditional version of Fatou's lemma. (f) Show that $\int_{B_r(0)} \|y\|_2^{2-n}\, dy = S_{n-1} r^2/2 < \infty$, where $B_r(0)$ is the n-dimensional ball with radius r centered the origin and S_{n-1} is the surface area of the $(n-1)$-dimensional unit sphere, and use an upper estimate for the continuous density of W_t. (g) Use Doob's convergence theorem for non-negative supermartingales. (h) Show that there is a constant c_n such that the continuous density of W_t outside $B_{\|x\|/2}(x)$ is bounded by $c_n/\|x\|_2^n$, uniformly for $t > 0$.

Remark: If $n = 3$, then M is the inverse three-dimensional Bessel process.

Remark: This problem shows that the stochastic exponential $\mathcal{E}(Y)$ of the continuous local martingale

$$Y_t := (2-n) \int_0^t \frac{W_s\, dW_s}{\|W_s\|_2^2}, \qquad t \geq 0,$$

is a strict local martingale, hence Kazamaki's criterion and Novikov's (local) criterion cannot be satisfied. Furthermore, for $\|x\|_2 = 1$ and (at least large) $T > 0$, the random variable M_T cannot be used to define an equivalent probability measure on $(\Omega, \mathcal{F}_T)$.