# 3. Problem Set for the Course Mathematical Finance 2: Continuous-Time Models

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**6. Problem:** *Conditional expectation involving independent random variables*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{G} \subset \mathcal{F}$ a sub-$\sigma$-algebra, $(S_1, \mathcal{S}_1)$ and $(S_2, \mathcal{S}_2)$ measurable spaces, $X : \Omega \to S_1$ and $Y : \Omega \to S_2$ random variables, and $F\colon S_1 \times S_2 \to \mathbb{R}$ an $\mathcal{S}_1 \otimes \mathcal{S}_2$-measurable function, which is bounded or non-negative. Suppose that $X$ is $\mathcal{G}$-measurable and $Y$ is independent of $\mathcal{G}$. Prove that

$$\mathbb{E}[F(X,Y)\,|\,\mathcal{G}] \overset{\text{a.s.}}{=} H(X), \qquad (*)$$

where $H(x) := \mathbb{E}[F(x,Y)]$ for all $x \in S_1$.

*Hint:* Show that the set

$$\mathcal{H} := \{F\colon S_1 \times S_2 \to \mathbb{R} \mid F \text{ is bounded and } \mathcal{S}_1 \otimes \mathcal{S}_2\text{-measurable satisfying } (*)\,\}$$

contains all $F$ of the form $F(x,y) = 1_A(x)1_B(y)$ with $A \in \mathcal{S}_1$ and $B \in \mathcal{S}_2$. Show that the monotone class theorem is applicable.

*Remark:* A version of this result is used to derive the Black–Scholes formula.

**7. Problem:** *First entrance time for closed sets, first contact time for open sets*
Let $X = \{X_t\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued process, adapted to a filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$. For $C \subset \mathbb{R}^d$ define $\tau := \inf\{\, t \geq 0 \,|\, X_t \in C\,\}$ with $\inf \varnothing := \infty$.

(a) Assume that $C$ is closed and $X$ has continuous paths. Show that the first entrance time $\tau$ of $X$ into $C$ is a stopping time.

(b) Give examples to show that $\tau$ might not be a stopping time if $C$ is open or if $X$ is just left-continuous.

(c) Assume that $C$ is open, $X$ has right-continuous paths and the filtration $\mathbb{F}$ is right-continuous. Show that the first contact time $\tau$ of $X$ with $C$ is a stopping time.

*Hints:* (a) Give a representation of $\{\tau \leq t\}$ involving open neighbourhoods of $C$. (b) Examples with $|\Omega| = 2$ suffice.

**8. Problem:**
Let $S$ be a geometric Brownian motion. For $\varepsilon > 0$ define $\tau_\varepsilon = \inf\{\, t \geq 0 \,|\, S_t \geq S_0 + \varepsilon\,\}$. Prove the following:

(a) $\lim_{\varepsilon\to\infty} \tau_\varepsilon \overset{\text{a.s.}}{=} \infty$,

(b) $\lim_{\varepsilon\searrow 0} \tau_\varepsilon \overset{\text{a.s.}}{=} 0$.

*Hints:* (a) Use the continuity of the paths. (b) Use Girsanov's theorem to change to a measure $\mathbb{Q} \sim \mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ such that $S$ has a drift of 1. Apply Doob's optional sampling theorem to $\{e^{-t}S_t\}_{t\in[0,T]}$ to show that $\mathbb{E}_{\mathbb{Q}}[\exp(-(\tau_\varepsilon \wedge T))] \geq S_0/(S_0 + \varepsilon)$.