4. Problem Set for the Course Mathematical Finance 2: Continuous-Time Models

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9. Problem: *Arbitrage possibility in the presence of fixed transaction costs*
Consider the Black–Scholes–Samuelson model with interest rate $r = 0$. Suppose that every rebalancing of the investment portfolio induces a fixed transaction cost $c \geq 0$. Show that for every $b > 0$ there is a self-financing, piecewise constant trading strategy φ generating almost surely the profit b during the time interval $[0, 1]$.

Hint: Adapt the definition of *self-financing* for piecewise constant strategies and generalize the example of an arbitrage strategy given in the lecture. Note that the initial investment, all rebalancings as well as the final liquidation of the stock position induce the cost c.

10. Problem: *Continuity of the Black–Scholes call option pricing formula*
The Black–Scholes call option pricing formula with stock price $s \geq 0$, time to maturity $\tau \geq 0$, strike $K \in \mathbb{R}$, riskless interest rate $r \in \mathbb{R}$ and volatility $\sigma \in \mathbb{R}$ is given in terms of the cumulative distribution function Φ of the standard normal distribution by

$$c_{\mathrm{BS}}(s,\tau,K,r,\sigma) = \begin{cases} s\,\Phi(d_1(s,\tau,K,r,\sigma)) - e^{-r\tau}K\,\Phi(d_2(s,\tau,K,r,\sigma)) & \text{if } s,\tau,K,|\sigma| > 0,\\ (s - e^{-r\tau}K)^+ & \text{otherwise,} \end{cases}$$

where

$$d_{1,2}(s,\tau,K,r,\sigma) := \frac{1}{|\sigma|\sqrt{\tau}}\biggl(\ln\frac{s}{K} + \Bigl(r \pm \frac{1}{2}\sigma^2\Bigr)\tau\biggr) \qquad \text{for } s,\tau,K,|\sigma| > 0.$$

Show that $c_{\mathrm{BS}}\colon [0,\infty)^2 \times \mathbb{R}^3 \to [0,\infty)$ is continuous.

Hints: Let X denote a standard normally distributed random variable. Verify and use the stochastic representation $c_{\mathrm{BS}}(s,\tau,K,r,\sigma) = \mathbb{E}\bigl[\bigl(s\exp(\sigma\sqrt{\tau}X - \frac{1}{2}\sigma^2\tau) - e^{-r\tau}K\bigr)^+\bigr]$.

11. Problem: *European options on stocks with known dividends*
Suppose a stock pays known dividends $\kappa_1, \ldots, \kappa_m \in \mathbb{R}$ at times $0 < t_1 < \cdots < t_m < T$. Let the capital gains be modeled by a stochastic process $\{G_t\}_{t\in[0,T]}$. Let $B_t = \exp(rt)$ for $t \in [0, T]$ denote the price process of the bank account unit with interest rate $r \in \mathbb{R}$. The stock price process $S = \{S_t\}_{t\in[0,T]}$ is modeled in case (I) as $S_t = G_t - D_t$, where

$$D_t = \sum_{j=1}^{m} \kappa_j e^{r(t-t_j)} 1_{[t_j,T]}(t)$$

denotes the value of the paid dividends up to time $t \in [0, T]$ with compounded interest, and in case (II) as $S_t = G_t + \hat{D}_t$, where

$$\hat{D}_t = \sum_{j=1}^{m} \kappa_j e^{-r(t_j-t)} 1_{[0,t_j)}(t)$$

denotes the value of the discounted future dividends at time $t \in [0,T]$. Let $\{C_t\}_{t\in[0,T]}$ and $\{P_t\}_{t\in[0,T]}$ denote the price processes of a European call and put option, respectively, with maturity $T > 0$ and strike $K \in \mathbb{R}$.

(a) If buy-and-hold strategies do not create arbitrage, show that the call–put parity holds, which in case (I) is given by

$$C_t - P_t \overset{\text{a.s.}}{=} S_t + D_t - (K + D_T)e^{-r(T-t)}, \qquad t \in [0,T],$$

and in case (II) by

$$C_t - P_t \overset{\text{a.s.}}{=} S_t - \hat{D}_t - Ke^{-r(T-t)}, \qquad t \in [0,T].$$

Assume now that the capital gains process G is modelled by

$$G_t = G_0 \exp\bigl(\sigma W_t + (\mu - \tfrac{1}{2}\sigma^2)t\bigr), \qquad t \in [0,T],$$

with appreciation rate $\mu \in \mathbb{R}$, volatility $\sigma > 0$, initial value $G_0 > 0$ and Brownian motion $\{W_t\}_{t\in[0,T]}$.

(b) Show that the arbitrage-free call and put option price processes are given in case (I) by

$$\begin{aligned} C_t &\overset{\text{a.s.}}{=} c_{\text{BS}}(S_t + D_t, T-t, K + D_T, r, \sigma), \\ P_t &\overset{\text{a.s.}}{=} p_{\text{BS}}(S_t + D_t, T-t, K + D_T, r, \sigma) \end{aligned}$$

and in case (II) by

$$\begin{aligned} C_t &\overset{\text{a.s.}}{=} c_{\text{BS}}(S_t - \hat{D}_t, T-t, K, r, \sigma), \\ P_t &\overset{\text{a.s.}}{=} p_{\text{BS}}(S_t - \hat{D}_t, T-t, K, r, \sigma) \end{aligned}$$

for all $t \in [0,T]$, where the Black–Scholes call option price formula c_{BS} is explicitly given in the previous problem and, correspondingly, the continuous put option price formula $p_{\text{BS}}\colon [0,\infty)^2 \times \mathbb{R}^3 \to [0,\infty)$ is given via call–put parity by

$$\begin{aligned} &p_{\text{BS}}(s,\tau,K,r,\sigma) \\ &\quad = c_{\text{BS}}(s,\tau,K,r,\sigma) + e^{-r\tau}K - s \\ &\quad = \begin{cases} e^{-r\tau}K\,\Phi(-d_2(s,\tau,K,r,\sigma)) - s\,\Phi(-d_1(s,\tau,K,r,\sigma)) & \text{if } s,\tau,K,|\sigma| > 0, \\ (e^{-r\tau}K - s)^+ & \text{otherwise.} \end{cases} \end{aligned}$$

(c) Calculate the replicating trading strategy and the *gamma* in both cases.

12. Problem: *Prices and Greeks of European options using implied volatility*
Consider the Black–Scholes–Samuelson model with stock price process $\{S_t\}_{t\geq 0}$ satisfying $S_0 = 100$ and interest rate $r = 3\%$. Assume that a European put option with maturity $T_{\text{p}} = 0.5$ and strike $K_{\text{p}} = 95$ is traded for $P_0 = 4$ at time 0.

(a) Calculate the implied volatility σ.

(b) Derive the replicating trading strategy and the *gamma* of a European put option and determine the specific values at time 0 for the above parameters.

(c) Calculate (for both cases of the previous problem) the price at time 0 of a call option with maturity $T_{\text{c}} = 1.5$ and strike $K_{\text{c}} = 105$, taking into consideration that there is a dividend of $\kappa = 5$ at time $t = 1$.

Hint: (a) Determine σ numerically. (c) Which σ do you take? What is your justification?