6. Problem Set for the Course Mathematical Finance 2: Continuous-Time Models

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15. Problem: *Class DL: Properties and examples*
An adapted right-continuous process $X = \{X_t\}_{t\geq 0}$ is said to be of class DL, if the family $\{X_\tau\}_{\tau\in\mathcal{T}_{[0,b]}}$ is uniformly integrable for every bound $b > 0$, where $\mathcal{T}_{[0,b]}$ denotes the set of all stopping times $\tau\colon \Omega \to [0, b]$. Show the following:

(a) The set of all adapted right-continuous process of class DL is a vector space.

(b) A right-continuous submartingale X is of class DL if

  (i) $X_t \geq 0$ almost surely for all $t \geq 0$ or

  (ii) X can be represented as $X_t = M_t + A_t$ for all $t \geq 0$, where $\{M_t\}_{t\geq 0}$ is a right-continuous martingale and $\{A_t\}_{t\geq 0}$ with $A_0 = 0$ is increasing (with $\mathbb{E}[A_t] < \infty$ for all $t \geq 0$).

Hint: (b) Use Doob's optional sampling theorem and the uniform integrability property of conditional expectations.

16. Problem: *Adding null sets to filtrations*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ a filtration of $\mathcal{F}$. Define

$$\mathcal{G}_t = \{G \in \mathcal{F} \mid \text{There exists } F \in \mathcal{F}_t \text{ with } \mathbb{P}(F \triangle G) = 0\,\}, \qquad t \geq 0,$$

and $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$. Show the following:

(a) $\mathbb{G}$ is a filtration and its σ-algebras contain all null sets of $\mathcal{F}$.

(b) If $\mathbb{F}$ is right-continuous, then $\mathbb{G}$ is right-continuous.

(c) If $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$, then $\mathbb{E}[X \mid \mathcal{F}_t] \overset{\text{a.s.}}{=} \mathbb{E}[X \mid \mathcal{G}_t]$ for all $t \geq 0$.

Hint: (b) For $G \in \mathcal{G}_{t+}$ and $n \in \mathbb{N}$ take $F_n \in \mathcal{F}_{t+1/n}$ with $\mathbb{P}(F_n \triangle G) = 0$. Consider $F := \limsup_{n\to\infty} F_n = \bigcap_{m\in\mathbb{N}} \bigcup_{n\geq m} F_n$.

Remark: Part (c) implies that $\mathbb{F}$-martingales are also $\mathbb{G}$-martingales, the same applies to sub- and supermartingales.

17. Problem: *A non-negative supermartingale without a Doob–Meyer decomposition*
Consider the open unit interval $\Omega = (0, 1)$ with Borel σ-algebra $\mathcal{F} = \mathcal{B}_{(0,1)}$ and Lebesgue–Borel measure $\mathbb{P}$. For $t \geq 0$ and $\omega \in \Omega$ define

$$X_t(\omega) = \begin{cases} \frac{1}{1-t} 1_{[0,\omega)}(t) & \text{if } t \in [0, 1), \\ 0 & \text{if } t \geq 1, \end{cases}$$

and

$$\mathcal{F}_t = \begin{cases} \{F \in \mathcal{F} \mid F \subset (0, t] \text{ or } F^{\mathrm{c}} \subset (0, t]\} & \text{if } t \in [0, 1), \\ \mathcal{F} & \text{if } t \geq 1. \end{cases}$$

Show the following:

(a) $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration.

(b) $X = \{X_t\}_{t\geq 0}$ is a non-negative, $\mathbb{F}$-adapted process with càdlàg paths.

(c) $\{X_t\}_{t\in[0,1)}$ is a martingale and X is a supermartingale.

(d) $\{X_t\}_{t\in[0,1)}$ is not uniformly integrable (hence X is not of class DL).

(e) X does not have a Doob–Meyer decomposition.

Hints: (c) Show for $s \leq t$ in $[0,1)$ that $X_t(\omega) = \frac{1-s}{1-t} X_s(\omega) 1_{[0,\omega)}(t)$ and $\mathbb{P}((t,1)\,|\,\mathcal{F}_s)(\omega) = \frac{1-t}{1-s} 1_{(s,1)}(\omega)$. (e) If there exists a decomposition $X = M - A$, then $X_t \overset{\text{a.s.}}{=} \mathbb{E}[A_1\,|\,\mathcal{F}_t]$ for all $t \in [0,1)$, contradicting (d).

18. Problem: *A uniformly integrable supermartingale without Doob–Meyer decomp.* Consider the open unit square $\Omega = (0,1)^2$ with Borel σ-algebra $\mathcal{G} = \mathcal{B}_{(0,1)^2}$ and two-dimensional Lebesgue–Borel measure $\mathbb{P}$. For $t \geq 0$ and $(\omega_1,\omega_2) \in \Omega$ define

$$Y_t(\omega_1,\omega_2) = \begin{cases} 1 & \text{if } t < \omega_1, \\ \frac{1}{1+\omega_1-t} & \text{if } t \in [\omega_1, \omega_1+\omega_2), \\ 0 & \text{if } t \geq \omega_1+\omega_2, \end{cases}$$

and

$$\mathcal{G}_t = \Big\{ G \in \mathcal{G} \;\Big|\; G_{\omega_1} \in \mathcal{F}_{t-\omega_1} \text{ for all } \omega_1 \in (0,t] \\ \text{and } G \text{ or } G^{\mathrm{c}} \text{ is a subset of } (0,t]\times(0,1) \Big\},$$

where $G_{\omega_1} = \{\omega_2 \in (0,1)\,|\,(\omega_1,\omega_2)\in G\}$ denotes the section of G at ω_1 and $\{\mathcal{F}_t\}_{t\geq 0}$ is the filtration given in the previous problem. Show the following:

(a) $\mathbb{G} = \{\mathcal{G}_t\}_{t\geq 0}$ is a right-continuous filtration.

(b) $Y = \{Y_t\}_{t\geq 0}$ is a non-negative, $\mathbb{G}$-adapted process with càdlàg paths.

(c) Y is a supermartingale satisfying for all $s \leq t$ in $[0,\infty)$ and almost all $\omega = (\omega_1,\omega_2)$

$$\mathbb{E}[Y_t\,|\,\mathcal{G}_s](\omega) = \begin{cases} \frac{2-t}{1-s}\, Y_s(\omega)\, 1_{[0,\omega_1)}(s) & \text{if } 1+s < t < 2, \\ Y_s(\omega)\, 1_{[0,1+\omega_1)}(t) & \text{otherwise.} \end{cases} \tag{$*$}$$

(d) Y is uniformly integrable.

(e) Y is not of class DL.

(f) Y does not have a Doob–Meyer decomposition.

Hints: Solve the previous problem first. (c) Show the $\mathcal{G}_s$-measurability of the right-hand side of $(*)$. To verify the integral condition for every $G \in \mathcal{G}_s$, use Fubini's theorem to integrate w.r.t. ω_2 first. (d) Show that $\mathbb{E}[Y_t 1_{\{Y_t\geq M\}}] \leq 1/M$ for all $M > 1$ and $t \geq 0$. (e) For $M > 1$ consider $\tau_M := \min\{2, \inf\{t\geq 0\,|\,Y_t \geq M\}\}$ and show that $\tau_M(\omega_1,\omega_2) = 1+\omega_1 - 1/M$ if $\omega_2 \geq (M-1)/M$ and $Y_{\tau_M}(\omega_1,\omega_2) = M 1_{[(M-1)/M,1)}(\omega_2)$. (f) If there exists a decomposition $Y = M - A$, then $Y_{\tau_M} \leq \mathbb{E}[A_2\,|\,\mathcal{G}_{\tau_M}]$ almost surely for all $M > 1$, contradicting (e).