7. Problem Set for the Course Mathematical Finance 2: Continuous-Time Models

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19. Problem: *Discussion of uniqueness in the Doob–Meyer decomposition*
On $\Omega = \{-1, 1\}$ with $\mathcal{F} = \mathcal{P}(\Omega)$ consider $\mathbb{P}$ with $p := \mathbb{P}(\{1\}) \in (0, \frac{1}{2})$ and the right-continuous the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ with $\mathcal{F}_t = \{\varnothing, \Omega\}$ for $t \in [0, 1)$ and $\mathcal{F}_t = \mathcal{F}$ for $t \geq 1$. Define $X_t(\omega) = \omega 1_{[1,\infty)}(t)$ for all $t \geq 0$ and $\omega \in \Omega$.

(a) Show that $X = \{X_t\}_{t\geq 0}$ is a càdlàg supermartingale of class D.

(b) Show that, for every constant $c \in [\frac{p}{1-p}, 1]$, there is a decomposition $X = M - A$ into a càdlàg martingale M and a corresponding increasing càdlàg process A with

$$M_t(\omega) = c\,\frac{1 + \omega - 2p}{2p} 1_{[1,\infty)}(t), \qquad t \geq 0,\, \omega \in \Omega.$$

(c) Determine all $c \in [\frac{p}{1-p}, 1]$ for which the increasing process A is natural.

20. Problem: *A discrete analogue of Girsanov's theorem*
On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let $X : \Omega \to \mathbb{R}^d$ be a normally distributed random vector with expectation vector $\mu \in \mathbb{R}^d$ and covariance matrix $C \in \mathbb{R}^{d\times d}$, i.e., $\mathbb{P}X^{-1} = \mathcal{N}(\mu, C)$. For $\xi \in \mathbb{R}^d$ define the tilted measure by

$$\mathbb{P}_\xi(A) = \mathbb{E}\big[1_A \exp\big(\langle \xi, X - \mu\rangle - \tfrac{1}{2}\langle \xi, C\xi\rangle\big)\big], \qquad A \in \mathcal{F}.$$

(a) Show that $\mathbb{P}_\xi$ is a probability measure on $(\Omega, \mathcal{F})$.

(b) Show that $\mathbb{P}_\xi X^{-1} = \mathcal{N}(\mu + C\xi, C)$.

Hints: (a) $\mathbb{P}\langle \xi, X-\mu\rangle^{-1} = \mathcal{N}(0, \langle \xi, C\xi\rangle)$ (b) Calculate the moment generating functions of $\mathbb{P}(X + C\xi)^{-1}$ and $\mathbb{P}_\xi X^{-1}$.

21. Problem: *A generalized Black–Scholes formula*
Let $X \sim \mathcal{N}(\mu, C)$ be a normally distributed random vector with expectation vector $\mu \in \mathbb{R}^d$ and covariance matrix $C \in \mathbb{R}^{d\times d}$. Given $\xi, \eta \in \mathbb{R}^d$, define

$$\sigma = \sqrt{\langle \xi - \eta, C(\xi - \eta)\rangle} = \sqrt{\operatorname{Var}(\langle \xi - \eta, X\rangle)}.$$

Show for all $a \in [0, \infty)$ and $b \in \mathbb{R}$ that

$$\mathbb{E}\Big[\Big(a \exp\big(\langle \eta, X - \mu\rangle - \tfrac{1}{2}\langle \eta, C\eta\rangle\big) - b \exp\big(\langle \xi, X - \mu\rangle - \tfrac{1}{2}\langle \xi, C\xi\rangle\big)\Big)^+\Big] = \begin{cases} a\,\Phi(d_1) - b\,\Phi(d_2) & \text{if } a, b, \sigma > 0, \\ (a - b)^+ & \text{otherwise,} \end{cases}$$

where Φ denotes the cumulative distribution function of the standard normal distribution and

$$d_{1,2} := \frac{1}{\sigma} \ln \frac{a}{b} \pm \frac{\sigma}{2} \qquad \text{for } a, b, \sigma > 0.$$

Hints: Consider the event D, that the difference is non-negative, and use the previous problem to calculate $\mathbb{P}_\eta(D)$ and $\mathbb{P}_\xi(D)$. Consider the boundary cases individually.

22. Problem: *Domestic price of a European call option on foreign equity*
Let $\{W_t^{\mathrm{d}}\}_{t\in[0,T]}$ be a d-dimensional Brownian motion under a domestic martingale measure $\mathbb{Q}_{\mathrm{d}}$, under which the exchange rate process is modelled by

$$Q_t = Q_0 \exp\bigl(\langle \sigma_Q, W_t^{\mathrm{d}}\rangle + (r_{\mathrm{d}} - r_{\mathrm{f}} - \tfrac{1}{2}\|\sigma_Q\|_2^2)t\bigr), \qquad t \in [0,T],$$

with $Q_0 > 0$ and the foreign equity price process by

$$S_t^{\mathrm{f}} = S_0^{\mathrm{f}} \exp\bigl(\langle \sigma_{S^{\mathrm{f}}}, W_t^{\mathrm{d}}\rangle + (r_{\mathrm{f}} + \tfrac{1}{2}\|\sigma_Q\|_2^2 - \tfrac{1}{2}\|\sigma_Q + \sigma_{S^{\mathrm{f}}}\|_2^2)t\bigr), \qquad t \in [0,T],$$

with $S_0^{\mathrm{f}} > 0$, where $r_{\mathrm{d}}, r_{\mathrm{f}} \in \mathbb{R}$ denote the domestic and foreign interest rate, respectively, and the volatility vectors $\sigma_Q, \sigma_{S^{\mathrm{f}}} \in \mathbb{R}^d$ are linearly independent. Show that the price process in domestic currency for a European call option on the foreign equity with maturity $T > 0$ and strike $K_{\mathrm{f}} \in \mathbb{R}$ in foreign currency is given by

$$\begin{aligned} C_t^{\mathrm{d}} &\overset{\text{a.s.}}{=} \mathbb{E}_{\mathbb{Q}_{\mathrm{d}}}\bigl[e^{-r_{\mathrm{d}}(T-t)} Q_T (S_T^{\mathrm{f}} - K_{\mathrm{f}})^+ \bigm| \mathcal{F}_t\bigr] \\ &\overset{\text{a.s.}}{=} \begin{cases} Q_t\bigl(S_t^{\mathrm{f}}\,\Phi(d_1(t)) - e^{-r_{\mathrm{f}}(T-t)}K_{\mathrm{f}}\,\Phi(d_2(t))\bigr) & \text{if } t \in [0,T) \text{ and } K_{\mathrm{f}} > 0, \\ Q_t\bigl(S_t^{\mathrm{f}} - e^{-r_{\mathrm{f}}(T-t)}K_{\mathrm{f}}\bigr)^+ & \text{if } t = T \text{ or } K_{\mathrm{f}} \le 0, \end{cases} \end{aligned}$$

where for the first case

$$d_{1,2}(t) := \frac{1}{\|\sigma_{S^{\mathrm{f}}}\|_2\sqrt{T-t}}\left(\ln\frac{S_t^{\mathrm{f}}}{K_{\mathrm{f}}} + \bigl(r_{\mathrm{f}} \pm \tfrac{1}{2}\|\sigma_{S^{\mathrm{f}}}\|_2^2\bigr)(T-t)\right).$$

Hint: Use Problem 6 and the previous one.

Remark: The same result follows by first calculating the value of the option in foreign currency (Black–Scholes formula) and using an additional no-arbitrage argument.