Name:

Mat.Nr.:

Bitte keinen Rotstift verwenden!

Finanzmathematik 2: Modelle in stetiger Zeit (Vorlesungsprüfung) 7. Oktober 2013 Privatdoz. Dr. Stefan Gerhold

90 Minuten

Unterlagen: ein handbeschriebener A4-Zettel sowie ein nichtprogrammierbarer Taschenrechner sind erlaubt

Anmeldung zur mündlichen Prüfung via TISS möglich. Wenn zu wenig Prüfungstermine online sind, bitte den Vortragenden Stefan Gerhold kontaktieren.

Bsp.	Max.	Punkte
1	10	
2	10	
3	8	
$\sum$	28	

Schriftlich:

AssistentIn:

Mündlich:

Gesamtnote:

1. Fix a time horizon $T \in (0, \infty)$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there is a Brownian motion $(W_t)_{0 \le t \le T}$. We take as filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ the one generated by W and augmented by the $\mathbb{P}$-nullsets in $\sigma(W_s; s \le T)$. Consider the Black Scholes model, where the bank account satisfies $B \equiv 1$, i.e. the interest rate $r \equiv 0$, and the undiscounted risky asset price is given by (10 Pkt.)

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 = 1,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. Moreover, $\mathbb{P}^* \sim \mathbb{P}$ denotes the unique equivalent martingale measure for the discounted price process $S = \frac{S}{B}$.

(a) Define a probability measure $\widehat{\mathbb{P}}$ by

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} := S_T. \tag{1}$$

(i) Argue why $(S_t)_{t \in [0,T]}$ qualifies as Radon-Nikodým derivative process.

(ii) Show that

$$\widehat{W}_t := W_t^* - \sigma t \tag{2}$$

is a $\widehat{\mathbb{P}}$-Brownian motion, where W^* denotes a $\mathbb{P}^*$-Brownian motion.

(ii) Use Bayes' formula to show that $\frac{1}{S}$ is a $\widehat{\mathbb{P}}$-martingale.

(b) Consider the process

$$\widehat{S}_t = \exp\left(-\sigma \widehat{W}_t - \frac{1}{2}\sigma^2 t\right),$$

where $\widehat{W}$ is a $\widehat{\mathbb{P}}$-Brownian motion, as specified in (2), and $\widehat{\mathbb{P}}$ denotes the measure defined in (1).

(i) What is the relation between $\widehat{S}$ and $\frac{1}{S}$.

(ii) Derive the SDE satisfied by $\widehat{S}$ under $\mathbb{P}$.

(iii) Consider a new market model under $\mathbb{P}$, where the bank account is given by $B \equiv 1$ and the stock price process by $\widehat{S}$. Does this model admit arbitrage? Is it complete?

(c) Let the call and put prices with maturity T and strike K and underlying process S be given by

$$C(T, K, S) = \mathbb{E}_{\mathbb{P}^*}\left[(S_T - K)^+\right], \quad P(T, K, S) = \mathbb{E}_{\mathbb{P}^*}\left[(K - S_T)^+\right].$$

Moreover $\widehat{C}(T, K, S)$ and $\widehat{P}(T, K, S)$ denote the corresponding prices under the measure $\widehat{\mathbb{P}}$. Prove the following identity

$$\frac{1}{K}C(T, K, S) = \widehat{P}\left(T, \frac{1}{K}, \frac{1}{S}\right).$$

2. Consider the setting of the Black Scholes model, as specified in the above example, however now the interest rate is no longer supposed to be 0, i.e., $r \ge 0$ and the bank account satisfies $B_t = e^{rt}$. An American digital put option with maturity $T > 0$ can be exercised at any time $t \in [0, T]$ at the choice of the option holder and yields the payoff $1_{[0,K]}(S_t)$ at time $t \in [0, T]$. The option holder wants to find a strategy which maximizes his payoff. (10 Pkt.)

(a) Consider the following possible situations at time t:

(i) $S_t \geq K$

(ii) $S_t < K$

In each case (i) and (ii), tell whether the option holder would choose to exercise the put option immediately or to wait.

(b) Show that the price at time 0 of an American digital put option with maturity T, strike K and initial stock price $S_0 = x > K$ is given by

$$P_d^a(0, x) = \mathbb{E}_{\mathbb{P}^*}\left[e^{-r\tau_K} 1_{\{\tau_K \leq T\}} | S_0 = x\right],$$

where

$$\tau_K = \inf\{t \geq 0 \,|\, S_t = K\}.$$

(c) It is known that in the Black-Scholes model the price of an American digital put $P_d^a(t, S_t)$ satisfies the PDE

$$rP_d^a(t, x) = \partial_t P_d^a(t, x) + rx\partial_x P_d^a(t, x) + \frac{1}{2}\sigma^2 x^2 \partial_{xx} P_d^a(t, x)$$

for $t \in [0, T)$ and $x > K$. Determine the boundary conditions $P_d^a(t, K)$, $0 \leq t < T$ and $P_d^a(T, x), x > K$ based on your answers in a).

(d) In the Black Scholes model the price at time t of an American digital put option with strike K can be computed via the following formula

$$P_d^a(t, x) = \frac{x}{K} N\left(\frac{-\ln\left(\frac{x}{K}\right) - \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}\right)$$
$$+ \left(\frac{x}{K}\right)^{-\frac{2r}{\sigma^2}} N\left(\frac{-\ln\left(\frac{x}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}\right), \quad x > K,$$

where N denotes the cumulative distribution function of the standard normal distribution.

Show that this formula is consistent with the boundary conditions derived in (c).

(e) Suppose that $\mu = r = 0$.

i) What is the probability to exercise the option in the interval $[0, T]$, if the initial stock price S_0 equals $x > K$ (and the holder of the option acts rationally)?

ii) In the case $r = 0$, is the price of the American digital put equal to the European digital put? Argue why this is true or false.

3. Consider a gap put with payoff (8 Pkt.)

$$(L - S_T)1_{\{S_T \leq K\}}.$$

(a) Draw the payoff function. For which values K and L does it take negative values?

(b) Decompose the option into a put- and a digital option.

(c) Find the price of the option at time 0 in the Black-Scholes model and determine a replicating portfolio.

(d) Determine the limits of the option price as $\sigma \to 0$ and $\sigma \to \infty$.