Name:

Mat.Nr.:

Bitte keinen Rotstift verwenden!

Finanzmathematik 2: Modelle in stetiger Zeit (Vorlesungsprüfung)

4. November 2013

Privatdoz. Dr. Stefan Gerhold

90 Minuten

Unterlagen: ein handbeschriebener A4-Zettel sowie ein nichtprogrammierbarer Taschenrechner sind erlaubt

Anmeldung zur mündlichen Prüfung via TISS möglich. Wenn zu wenig Prüfungstermine online sind, bitte den Vortragenden Stefan Gerhold kontaktieren.

Bsp.	Max.	Punkte
1	12	
2	8	
3	8	
$\sum$	28	

Schriftlich:

AssistentIn:

Mündlich:

Gesamtnote:

1. Fix a time horizon $T \in (0, \infty)$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there is a Brownian motion $(W_t)_{0 \le t \le T}$. We take as filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ the one generated by W and augmented by the $\mathbb{P}$-nullsets in $\sigma(W_s; s \le T)$. Consider the Black Scholes model, where the bank account satisfies $B \equiv 1$, i.e. the interest rate $r \equiv 0$, and the risky asset price is given by (12 Pkt.)

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 = 1,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. Moreover, $\mathbb{P}^* \sim \mathbb{P}$ denotes the unique equivalent martingale measure for the discounted price process $S = \frac{S}{B}$.

(a) Let H be a nonnegative $\mathcal{F}_T$-measurable payoff due at time T.

(i) Construct a probability measure $\widehat{\mathbb{P}} \sim \mathbb{P}^*$ such that

$$E_{\mathbb{P}^*}[H] = E_{\widehat{\mathbb{P}}}\left[\frac{H}{S_T}\right].$$

Specify in particular the candidate density process $(Z)_{0 \le t \le T}$ and show that it satisfies all necessary properties such that

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} := Z_T \tag{1}$$

defines an equivalent probability measure $\widehat{\mathbb{P}} \sim \mathbb{P}^*$.

(ii) Show that

$$\widehat{W}_t := W_t^* - \sigma t \tag{2}$$

is a $\widehat{\mathbb{P}}$-Brownian motion, where W^* denotes a $\mathbb{P}^*$-Brownian motion.

(iii) Use Bayes' formula to show that $\frac{1}{S}$ is a $\widehat{\mathbb{P}}$-martingale.

(b) Consider the process

$$\widehat{S}_t = \exp\left(-\sigma \widehat{W}_t - \frac{1}{2}\sigma^2 t\right), \tag{3}$$

where $\widehat{W}$ is a $\widehat{\mathbb{P}}$-Brownian motion, as specified in (2), and $\widehat{\mathbb{P}}$ denotes the measure defined in (1).

(i) Derive the SDE satisfied by $\widehat{S}$ under $\widehat{\mathbb{P}}$.

(ii) What is the relation between $\widehat{S}$ and $\frac{1}{S}$?

(c) Consider a lookback call option with floating strike, whose payoff at time T is given by

$$H = \left(S_T - \alpha \min_{0 \le t \le T} S_t\right)^+, \quad \alpha \ge 1. \tag{4}$$

Show that its price at time 0 can be expressed by

$$\alpha E_{\widehat{\mathbb{P}}}\left[\left(\frac{1}{\alpha} - \min_{0 \le t \le T} \widehat{S}_t\right)^+\right],$$

where $\widehat{S}_t$ is given by (3) and $\widehat{\mathbb{P}}$ denotes the measure defined in (1).
Hint: Use the reflection principle for a Brownian motion with drift: If $X_t = bt + c\widehat{W}_t$, $b, c \in \mathbb{R}$ and $\widehat{W}$ a $\widehat{\mathbb{P}}$-Brownian motion, then we have for all $x \in \mathbb{R}$

$$\widehat{\mathbb{P}}\left[\max_{0\leq t\leq T} X_t - X_T \leq x\right] = \widehat{\mathbb{P}}\left[-\min_{0\leq t\leq T} X_t \leq x\right].$$

This means that $\max_{0\leq t\leq T} X_t - X_T$ and $-\min_{0\leq t\leq T} X_t$ have the same law.

(d) Let $\alpha = 1$ in (4). Show that the price of an American lookback call is the same as the European counterpart.

2. Consider the setting of the Black Scholes model, as specified in the above example, however now the interest rate is no longer supposed to be 0, i.e., $r \geq 0$ and the bank account satisfies $B_t = e^{rt}$. Moreover, consider a zero-strike Asian call whose payoff at time T is (8 Pkt.)

$$H = \frac{1}{T}\int_0^T S_u du.$$

(a) Suppose at time t we have $S_t = x \geq 0$ and $Y_t = \int_0^t S_u du = y \geq 0$. Use the fact that $(e^{-ru}S_u)_{u\in[0,T]}$ is a martingale under $\mathbb{P}^*$ to compute

$$e^{-r(T-t)}\mathbb{E}_{\mathbb{P}^*}\left[\frac{1}{T}\int_0^T S_u du|\mathcal{F}_t\right]$$

and denote this by $v(t, x, y)$.

(b) Determine explicitly the process $\Delta_t = v_x(t, S_t, Y_t)$ and observe that it is not random.

(c) Use Itô's formula to show that if you begin with initial capital $X_0 = v(0, S_0, 0)$ and at each time you hold Δ_t shares of S, investing or borrowing at the interest rate r in order to do this, then at time T the value of your portfolio will be

$$X_T = \frac{1}{T}\int_0^T S_u du.$$

3. Consider a gap call with payoff (8 Pkt.)

$$(S_T - L)1_{\{S_T > K\}}.$$

(a) Draw the payoff function. For which values K and L does it take negative values?

(b) Decompose the option into an asset-or-nothing binary option and a cash-or-nothing binary option.

(c) Find the price of the option at time 0 in the Black-Scholes model and determine a replicating portfolio.

(d) Determine the limits of the option price as $\sigma \to 0$ and $\sigma \to \infty$.