Mathematical Finance 2: Continuous-Time Models
Exercise sheet 1

March 12, 2013

1. Consider a discrete-time model with finite time horizon T defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{n\in\{0,\dots,T\}}, \mathbb{P})$ with stock price process S and bank account B whose initial value satisfies $B_0 = 1$. A strategy $\phi = (\phi^{(0)}, \phi^{(1)})$, that is, a predictable process[1] taking values in $\mathbb{R}^2$ is called *self-financing* if, for every $n \in \{1, \dots, T\}$, the value of the portfolio given by

$$V_n(\phi) := \phi_n^{(0)} B_n + \phi_n^{(1)} S_n$$

satisfies

$$V_n(\phi) = V_0(\phi) + \sum_{j=1}^{n} \phi_j^{(0)}(B_j - B_{j-1}) + \sum_{j=1}^{n} \phi_j^{(1)}(S_j - S_{j-1}).$$

a) Prove that a strategy ϕ is self-financing if and only if

$$\phi_n^{(0)} B_n + \phi_n^{(1)} S_n = \phi_{n+1}^{(0)} B_n + \phi_{n+1}^{(1)} S_n$$

for all $n \in \{0, \dots, T-1\}$. Give a verbal interpretation of this property.

b) Show that for any $\mathbb{R}$-valued predictable process $\phi^{(1)}$ and any $\mathcal{F}_0$-measurable random variable V_0, there exists a unique real-valued predictable process $\phi^{(0)}$ such that the strategy $\phi = (\phi^{(0)}, \phi^{(1)})$ is self-financing with $V_0(\phi) = V_0$.

2. Consider the multi-period Cox-Ross-Rubinstein binomial model with bank account $B_n = (1+r)^n$, $r > -1$, and stock price process S, where S evolves between two consecutive periods as

$$S_{n+1} = S_n Z_{n+1}, \quad n = 0, \dots, T-1, \quad S_0 > 0.$$

Here $(Z_n)_{n=1,\dots,T}$ are i.i.d random variables, taking only the two values u and d for $u > d > 0$ with probabilities

$$\mathbb{P}[Z_n = u] = p \quad \text{and} \quad \mathbb{P}[Z_n = d] = 1 - p, \quad p \in (0, 1).$$

[1] In the discrete time setting, this simply means that ϕ_n is $\mathcal{F}_{n-1}$-measurable for all $n \in \{1, \dots, T\}$.

Please turn over!

a) If $u \leq 1+r$ or $d \geq 1+r$, find an arbitrage opportunity, that is, a self-financing strategy ϕ with $V_0(\phi) = 0$, $V_T(\phi) \geq 0$ $\mathbb{P}$-a.s. and $\mathbb{P}[V_T(\phi) > 0] > 0$, where $V = \phi^{(0)}B + \phi^{(1)}S$ denotes the value process.

b) If $u > 1+r > d$, find an equivalent probability measure $\mathbb{Q} \approx \mathbb{P}$ such that the discounted price process $\frac{S}{B}$ is a $\mathbb{Q}$-martingale.

3. Let $W = (W_t)_{t \geq 0}$ be a standard n-dimensional Brownian motion and $\sigma \in \mathbb{R}^n$. In every one of the cases

a) $M_t = \sum_{i=1}^n \sigma_i W_{t,i}$,

b) $M_t = \|W_t\|_2^2 - nt$,

c) $M_t = \exp\left(\sum_{i=1}^n \sigma_i W_{t,i} - \frac{1}{2}\|\sigma\|_2^2 t\right)$

find a progressively measurable process $H : [0,\infty) \times \Omega \to \mathbb{R}^n$ with

$$\mathbb{E}\left[\int_0^t \|H_s\|_2^2 ds\right] < \infty, \quad t \geq 0,$$

such that

$$M_t \overset{\text{a.s.}}{=} \mathbb{E}[M_0] + \sum_{i=1}^n \int_0^t H_{s,i} dW_{s,i}\,.$$

Hint: Apply the n-dimensional Itô-formula.

4. Let $W = (W_t)_{t \geq 0}$ be a standard one-dimensional Brownian motion and $\sigma \in \mathbb{R}$. Give a direct proof that

$$S_t = S_0 \exp\left(\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t\right), \quad t \geq 0$$

is (up to indistinguishability) the unique strong solution of the stochastic differential equation

$$dS_t = S_t \mu dt + S_t \sigma dW_t, \quad t \geq 0,$$

with deterministic initial condition $S_0 > 0$.

Hint: For uniqueness, let $\widetilde{S}$ denote another solution of the SDE and apply the two-dimensional Itô-formula to the process $X_t = \frac{\widetilde{S}_t}{S_t}$ for $t \geq 0$.