Mathematical Finance 2: Continuous-Time Models
Exercise sheet 2

March 21, 2013

1. a) Define for $\mu \in \mathbb{R}$, $x, y \in \mathbb{R}$ and $\tau > 0$

$$p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left(-\frac{(y - x - \mu\tau)^2}{2\tau}\right).$$

Show that p is the transition probability density function of a Brownian motion with drift

$$X_t = \mu t + W_t.$$

This means to show that for bounded measurable functions f and $t > s$, we have

$$E[f(X_t) \mid \mathcal{F}_s] = g(X_s)$$

with $g(x) = \int_{-\infty}^{\infty} f(y)p(t - s, x, y)dy$.

b) Show that the transition probability density function of a geometric Brownian motion

$$S_t = S_0 e^{\sigma W_t + \nu t}$$

is given by

$$p(\tau, x, y) = \frac{1}{\sigma y \sqrt{2\pi\tau}} \exp\left(-\frac{(\log(y/x) - \nu\tau)^2}{2\sigma^2\tau}\right),$$

where $x, y, \tau > 0$.

2. Let $T > 0$ and let $(W_t)_{t\in[0,T]}$ denote a standard Brownian motion. Consider $S_t = e^{W_t - \frac{1}{2}t}$ as model for the stock price. Suppose that the interest rate r equals 0. Define

$$Y_t := \int_0^t \sqrt{\frac{1}{T - u}} dW_u$$

and the stopping time

$$\tau := \inf\left\{t \in [0, T) \,\middle|\, Y_t = 1\right\} \wedge T.$$

Please turn over!

a) Show that $0 < \tau < T$ almost surely. *Hint:* It is useful to first show that $(Y_{T-Te^{-t}})_{t\geq 0}$ is a Brownian motion.

b) Let $V_0 = 0$ and define the self-financing trading strategy

$$\varphi_t^{(1)} = \frac{1}{S_t}\sqrt{\frac{1}{T-t}}1_{\{t\leq\tau\}}, \quad \varphi_t^{(0)} = Y_{t\wedge\tau} - \sqrt{\frac{1}{T-t}}1_{\{t\leq\tau\}}.$$

Compute the value of the wealth process V_t and conclude using a) that $V_T = 1$ almost surely.

c) Prove that V is not bounded from below.

3. For a European call option expiring at time T with strike price K the Black-Scholes price at time t, if the time-t stock price is x, is given by

$$c(t,x) = xN\left(d_1(T-t,x)\right) - Ke^{-r(T-t)}N\left(d_2(T-t,x)\right), \tag{1}$$

where

$$d_1(\tau,x) = \frac{\ln\left(\frac{x}{K}\right) + (r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \qquad d_2(\tau,x) = d_1(\tau,x) - \sigma\sqrt{\tau},$$

and N is the standard Gaussian distribution function. Moreover, in the sequel $n(y)$ denotes the standard Gaussian probability density $n(y) = \frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$.

a) Verify

$$Ke^{-r(T-t)}n\left(d_2(T-t,x)\right) = xn\left(d_1(T-t,x)\right).$$

b) Show that

$$\partial_x c(t,x) = N(d_1(T-t,x)).$$

This is the so-called *delta* of the option.

c) Show that

$$\partial_{xx}c(t,x) = \frac{n(d_1(T-t,x))}{x\sigma\sqrt{T-t}},$$

which is the so-called *gamma* of the option.

d) Show that

$$\partial_t c(t,x) = -rKe^{-r(T-t)}N(d_2(T-t,x)) - \frac{x\sigma}{2\sqrt{T-t}}n(d_1(T-t,x)),$$

which is the so-called *theta* of the option.

Please turn over!

e) Use these formulas to show that $c(t, x)$ satisfies the Black-Scholes partial differential equation:

$$\partial_t c(t,x) + rx\partial_x c(t,x) + \frac{1}{2}\sigma^2 x^2 \partial_{xx} c(t,x) = rc(t,x), \quad 0 \le t < T, \quad x > 0.$$

4. Consider a stock which is modeled by

$$dS_t = r(t)S_t dt + \sigma(t)S_t dW_t^*, \quad S_0 > 0,$$

where $r(t)$ and $\sigma(t)$ are deterministic functions of t and W^* is a Brownian motion under the martingale measure $\mathbb{P}^*$. Let $T > 0$ be given and consider a European call whose value at time zero is

$$c(S_0, 0) = \mathbb{E}_{\mathbb{P}^*}\left[e^{-\int_0^T r(t)dt}(S(T) - K)^+\right].$$

a) Show that S_T is of the form $S_0 e^X$ where X is a normal random variable and determine the mean and the variance of X.

b) Let $BS(x, r, \sigma)$ denote the Black Scholes price at time 0 of a European call with strike K and maturity T when the underlying stock has initial value x and constant volatility σ and the interest rate r is constant. Show that

$$c(S_0, 0) = BS\left(S_0, \frac{1}{T}\int_0^T r(t)dt, \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)dt}\right).$$