Mathematical Finance 2: Continuous-Time Models
Exercise sheet 3

April 11, 2013

1. Let W be a one-dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let h be a Borel measurable function $h : \mathbb{R} \to \mathbb{R}$. Fix $T > 0$ and let $t \in [0, T]$ and $x \in \mathbb{R}$. Assume that $\mathbb{E}_{\mathbb{P}}[|h(W_T)||W_t = x] < \infty$ for all t and x and define the function

$$u(x,t) := \mathbb{E}_{\mathbb{P}}\left[e^{-r(T-t)}h(W_T)|W_t = x\right].$$

Prove that $u(t, x)$ satisfies the following PDE

$$-\frac{\partial u(x,t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(x,t)}{\partial x^2} - ru(x,t),$$

with terminal condition $u(x, T) = h(x)$ for $x \in \mathbb{R}$.

2. Consider the Black-Scholes model and let $c(t, x)$ denote the Black-Scholes price at time t of a European call option with maturity T and strike K, when the time-t stock price is x. Consider a dynamic portfolio $\Phi_t = (\Phi_t^1, \Phi_t^2)$ given by

$$\Phi_t = (\partial_x c(t, S_t), -1),$$

where $\Phi_t^1 = \partial_x c(t, S_t)$ and $\Phi_t^2 = -1$ stand for the number of shares of the stock and the number of call options held at time t respectively. The wealth of this strategy at time t thus equals

$$V_t = \Phi_t^1 S_t + \Phi_t^2 c(t, S_t) = \partial_x c(t, S_t)S_t - c(t, S_t).$$

a) Show that $V_T = K1_{\{S_T > K\}}$ and thus corresponds to a cash-or-nothing digital call. Compute also V_0.

b) Compute the price at time 0 of the cash-or-nothing digital call (with payoff given by $K1_{\{S_T > K\}}$) using the risk neutral valuation formula. What is the relationship to the derivative of $c(0, S_0)$ with respect to K.

Please turn over!

c) As shown in the lecture the strategy Φ is not self-financing. Why does V_0 correspond nevertheless to the price of the cash-or-nothing digital call?

3. Let $T > 0$ be fixed and let $\mu \in \mathbb{R}$, $\sigma, r \in \mathbb{R}_+$. Let W be a one dimensional Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider the process

$$X_t := \exp\left(\frac{r-\mu}{\sigma}W_t - \frac{1}{2}\frac{(r-\mu)^2}{\sigma^2}t\right).$$

Show that X is a martingale with $\mathbb{E}_{\mathbb{P}}[X_T] = 1$.

4. Let $T > 0$ be fixed and let r be the riskless interest rate for continuous compounding. Denote by C_t the price at time t of a European call option on the underlying stock S with payoff $(S_T - K)^+$. Show that for every $t \in [0, T]$

$$(S_t - e^{-r(T-t)}K)^+ \leq C_t \leq S_t.$$

These are general arbitrage bounds that hold in any market that is free of arbitrage. Thus do not use any particular model structure, like the Black-Scholes model. Does a similar relation hold for the intrinsic value $(S_t - K)^+$ of the option?
Hint: Prove the inequalities by contradiction. Construct arbitrage portfolios from bond, stock, and option.

5. Consider the Black-Scholes model and let $c(t, x)$ denote the Black-Scholes price at time t of a European call option with maturity T and strike K, when the time-t stock price equals x. Define the *elasticity*, i.e., the relative change of the option's price as the stock prices moves by

$$\eta_t^c := \frac{\partial_x c(t, S_t) S_t}{c(t, S_t)}.$$

a) Show that $\eta_t^c > 1$. Moreover, show that, if the stock price changes, the absolute change in the option price is smaller than the absolute change in the stock price.

b) Show that the dynamics of the option price under the risk neutral measure $\mathbb{P}^*$ satisfy

$$dc(t, S_t) = c(t, S_t)(rdt + \sigma\eta_t^c dW_t^*),$$

where W^* is a standard Brownian motion under $\mathbb{P}^*$.