Mathematical Finance 2: Continuous-Time Models
Exercise sheet 4

April 18, 2013

1. Compute the limit of the Black-Scholes formula for $\sigma \to 0$, $\sigma \to \infty$, $T \to 0$ and $T \to \infty$ and give an economic interpretation.

2. Consider the Black-Scholes model with $r = 0$ and denote the price of a European call option at time 0 with maturity T and strike K by C_0. Show Dupire's formula that is

$$\frac{\partial C_0}{\partial T} = \frac{\sigma^2 K^2}{2} \frac{\partial^2 C_0}{\partial K^2}.$$

Hint: In order to compute the second derivative with respect to K it can be helpful to write C_0 as

$$C_0 = \int_K^\infty (s - K) p(s) ds,$$

where p denotes the transition probability density function of S_T (which is log-normal in the case of the Black-Scholes model).

3. Consider the Black-Scholes model and let $c(t, x)$ denote the Black-Scholes price at time t of a European call option with maturity T and strike K, when the time-t stock price is x. Prove the following representation of the Black-Scholes formula

$$c(t, x) = (x - K)^+ + x \int_0^a \varphi \left(\frac{\ln(x/K)}{y} + \frac{y}{2} \right) dy,$$

where $a = \sigma\sqrt{T - t}$ and φ denotes the standard normal density.

4. Consider the Black-Scholes model on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ with finite time horizon $T^* > 0$ and suppose that the filtration is generated by the underlying Brownian motion. Let X be some $\mathcal{F}_{T^*}$-measurable claim such that $\mathbb{E}_{\mathbb{P}^*}[X^2] < \infty$, where $\mathbb{P}^*$ denotes the spot martingale measure. Consider a future contract on X with time of delivery $T^* > 0$. Following Björk (2004), we

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define a future contract as financial asset with a price process Π^f and a dividend process D satisfying

$$\begin{aligned} D_t &= f_X(t, T^*), \\ f_X(T^*, T^*) &= X, \\ \Pi_t^f &= 0, \quad \forall t \leq T^*, \end{aligned}$$

where $(f_X(t, T^*))_{t \in [0, T^*]}$ denotes the future price.

a) Prove that $(f_X(t, T^*))_{t \in [0, T^*]}$ given by

$$f_X(t, T^*) := \mathbb{E}_{\mathbb{P}^*}\left[X | \mathcal{F}_t\right]$$

is the unique process such that $f_X(T^*, T^*) = X$ and the discounted gains process V of holding a future contract, that is,

$$V_t = B_t^{-1} \Pi_t^f + \int_0^t B_s^{-1} dD_s = \int_0^t B_s^{-1} df_X(s, T^*)$$

is a $\mathbb{P}^*$ martingale.

b) How do the dynamics of the future price $f_S(t, T^*)$ of the stock S look like under $\mathbb{P}^*$ and $\mathbb{P}$?

5. Consider Black's future model. Prove that the future strategy given by

$$\phi_t^1 = \frac{\partial c^f(f_t, T-t)}{\partial f}, \quad \phi_t^2 = e^{-rt} c^f(f_t, T-t),$$

where $c^f(f, \tau)$ denotes Black's future formula and (f_t) the future price process, is an admissible self-financing future strategy.