Mathematical Finance 2 : Continuous-Time Models
Exercise sheet 5

April 25, 2013

1. Consider a one-dimensional Itô-process model with finite time horizon $T > 0$, where the stock price dynamics are given by

$$dS_t = S_t \mu_t dt + S_t \sigma_t dW_t, \quad S_0 = s > 0.$$

Here, W denotes a standard one-dimensional Brownian motion with respect to some filtration $(\mathcal{F}_t)_{0 \le t \le T}$ and $\sigma \ge 0$ and μ are assumed to be real-valued predictable processes. The bank account is modeled by

$$dB_t = B_t r dt, \quad B_0 = 1,$$

where r is the constant interest rate. Suppose furthermore that there exists a constant $\sigma^* > 0$ such that the inequality $\sigma_t \le \sigma^*$ holds for every $t \in [0, T]$ a.s. Consider now a self-financing strategy $(\phi^{(1)}, \phi^{(2)})$ defined via $\phi_t^{(1)} = N(d_1(S_t, t))$ and $V_0(\varphi) \ge c(0, S_0)$, where c corresponds to the Black-Scholes price and $\phi_t^{(1)}$ to the Black-Scholes delta with volatility σ^* and some strike K. Prove that $V_T(\phi) \ge (S_T - K)^+$ a.s.

2. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $X : \Omega \to \mathbb{R}^d$ be a normally distributed random vector with expectation $\mu \in R^d$ and covariance matrix $C \in \mathbb{R}^{d \times d}$, i.e., the law $\mathbb{P}X^{-1}$ of X is $\mathbb{P}X^{-1} = N(\mu, C)$. For $\xi \in \mathbb{R}^d$ define the tilted measure by

$$\mathbb{P}(A) = \mathbb{E}\left[1_A \exp\left(\langle \xi, X - \mu \rangle - \frac{1}{2}\langle \xi, C\xi \rangle\right)\right], A \in \mathcal{F},$$

where $\langle \cdot, \cdot \rangle$ stands for the Euclidean scalar product on $\mathbb{R}^d$.

a) Show that $\mathbb{P}_\xi$ is a probability measure on $(\Omega, \mathcal{F})$.

b) Show that $\mathbb{P}_\xi X^{-1} = N(\mu + C\xi, C)$.

Hints : ad b) Calculate the moment generating functions of $\mathbb{P}(X + C\xi)^{-1}$ and $\mathbb{P}_\xi X^{-1}$.

Please turn over !

3. *A generalized Black-Scholes formula :*
Let $X \sim N(\mu, C)$ be a normally distributed random vector with expectation vector $\mu \in \mathbb{R}^d$ and covariance matrix $C \in \mathbb{R}^{d\times d}$. Given $\xi, \eta \in \mathbb{R}^d$, define

$$\sigma := \sqrt{\langle \xi - \eta, C(\xi - \eta)\rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes again the Euclidean scalar product on $\mathbb{R}^d$. Show for all $a \in [0, \infty)$ and $b \in \mathbb{R}$ that

$$\mathbb{E}\left[\left(a\exp\left(\langle \eta, X-\mu\rangle - \frac{1}{2}\langle \eta, C\eta\rangle\right) - b\exp\left(\langle \xi, X-\mu\rangle - \frac{1}{2}\langle \xi, C\xi\rangle\right)\right)^+\right]$$
$$= \begin{matrix} aN(d_1) - bN(d_2), & \text{if } a, b, \sigma > 0, \\ (a-b)^+, & \text{otherwise,} \end{matrix}$$

where N denotes the cumulative distribution function of the standard normal distribution and

$$d_{1,2} := \frac{1}{\sigma}\ln\left(\frac{a}{b}\right) \pm \frac{\sigma}{2}, \quad a, b, \sigma > 0,$$

Hints : Consider the event D, that the difference is non-negative, and use the previous problem to calculate $\mathbb{P}_\xi(D)$ and $\mathbb{P}_\eta(D)$. Consider the boundary cases individually.

4. *Domestic price of a European call option on foreign equity :* Let $(W_t)_{t\in[0,T]}$ be a d-dimensional Brownian motion under a domestic martingale measure $\mathbb{P}^*$, under which the exchange rate process is modeled by

$$dQ_t = Q_t(r_d - r_f)dt + Q_t\langle \sigma_Q, dW_t\rangle, \quad Q_0 > 0$$

and the foreign equity price process by

$$S_t^f = S_0^f \exp\left(\langle \sigma_{S^f}, W_t\rangle + (r_f + \frac{1}{2}\|\sigma_Q\|^2 - \frac{1}{2}\|\sigma_Q + \sigma_{S^f}\|^2)t\right), t \in [0,T]$$

with $S_0^f > 0$, where r_d, r_f denote the domestic and foreign interest rate, respectively, and the volatility vectors $\sigma_Q, \sigma_{S^f} \in \mathbb{R}^d$ are linearly independent. Show that the price process in domestic currency for a European call option on the foreign equity with maturity $T > 0$ and strike $K^f \in \mathbb{R}$ in foreign currency is given by

$$C_t = \mathbb{E}_{\mathbb{P}^*}\left[e^{-r_d(T-t)}Q_T(S_T^f - K^f)^+ | \mathcal{F}_t\right]$$
$$= \begin{matrix} Q_t(S_t^f N(d_1) - e^{-r_f(T-t)}K^f N(d_2)), & \text{if } t \in [0,T), K^f > 0 \\ Q_t(S_t^f - e^{-r_f(T-t)}K^f)^+, & \text{if } t = T \text{ or } K^f \leq 0. \end{matrix}$$

Please turn over !

where for the first case

$$d_{1,2} := \frac{1}{\|\sigma_{S^f}\|\sqrt{T-t}} \left(\ln\left(\frac{S_t^f}{K^f} \right) + (r_f \pm \|\sigma_{S^f}\|^2)(T-t) \right).$$

Hint : Use the previous exercise.

5. Consider the cross-currency model from the lecture and let X be a contingent claim, which settles at time T and is denominated in the domestic currency. Let $\pi_t(X)$ denote the arbitrage-free price of X in the domestic currency and $\widetilde{\pi}_t(X)$ the arbitrage-free price of X at time t in units of the foreign currency. Prove that

$$\pi_t(X) = Q_t \widetilde{\pi}_t(X),$$

where Q denotes the exchange rate, which represents the domestic price of one unit of the foreign currency.