Mathematical Finance 2 : Continuous-Time Models
Exercise sheet 6

May 2, 2013

1. Let $(W_t)_{t\in[0,T]}$ be a d-dimensional Brownian motion under a domestic martingale measure $\mathbb{P}^*$, under which the exchange rate process is modeled by

$$dQ_t = Q_t(r_d - r_f)dt + Q_t\langle\sigma_Q, dW_t\rangle, \quad Q_0 > 0$$

and the foreign equity price process by

$$S_t^f = S_0^f \exp\left(\langle\sigma_{S^f}, W_t\rangle + \left(r_f + \frac{1}{2}\|\sigma_Q\|^2 - \frac{1}{2}\|\sigma_Q + \sigma_{S^f}\|^2\right)t\right), \quad t \in [0,T]$$

with $S_0^f > 0$, where r_d, r_f denote the domestic and foreign interest rate, respectively, and the volatility vectors $\sigma_Q, \sigma_{S^f} \in \mathbb{R}^d$ are linearly independent. Determine the price (in units of the domestic currency) of an equity-linked foreign exchange call, that is, an option with payoff (in units of domestic currency)

$$(Q_T - K)^+ S_T^f,$$

where K is a strike exchange rate expressed in domestic currency

a) by computing $e^{-r_d(T-t)}\mathbb{E}_{\mathbb{P}^*}\left[(Q_T - K)^+ S_T^f|\mathcal{F}_t\right]$.

b) by showing that

$$e^{-r_d(T-t)}\mathbb{E}_{\mathbb{P}^*}\left[(Q_T - K)^+ S_T^f|\mathcal{F}_t\right] = e^{-r_f(T-t)}Q_t\mathbb{E}_{\widetilde{\mathbb{P}}}\left[\left(S_T^f - K\frac{S_T^f}{Q_T}\right)^+\Big|\mathcal{F}_t\right]$$

and computing the right hand side by means of Exercise 5.3. Here, $\widetilde{\mathbb{P}}$ denotes the foreign martingale measure defined via

$$\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}^*} = e^{\langle\sigma_Q, W_T\rangle - \frac{1}{2}\|\sigma_Q\|^2 T}.$$

Please turn over !

2. Consider the cross-currency model from the lecture and let

$$h_{1,2}(q,t) = \frac{\ln\left(\frac{q}{K}\right) + (r_d - r_f \pm \frac{1}{2}\|\sigma_Q\|^2)t}{\|\sigma_Q\|\sqrt{t}}.$$

Prove that the strategy $\phi = (\phi^{(1)}, \phi^{(2)})$, with $\phi_t^{(1)} = e^{-r_f T} N(h_1(Q_t, T-t))$ and $\phi_t^{(2)} = -Ke^{-r_d T} N(h_2(Q_t, T-t))$ is a self-financing currency trading strategy (see Proposition 4.2.2 in Musiela, Rutkowski).

3. (Conditional Jensen inequality) Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a sub-σ-algebra $\mathcal{G} \subset \mathcal{F}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a convex function and X a random variable such that X and $g(X) \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Prove that

$$g(\mathbb{E}[X|\mathcal{G}]) \leq \mathbb{E}[g(X)|\mathcal{G}].$$

4. Let $T > 0$ be fixed and let $r \geq 0$ be the constant riskless interest rate for continuous compounding. Denote by C_t^a and P_t^a (C_t^e and P_t^e) the price at time t of American call and put options (European call and put options, respectively). Show by no arbitrage arguments that for every $t \in [0, T]$

a) $C_t^e = C_t^a$,

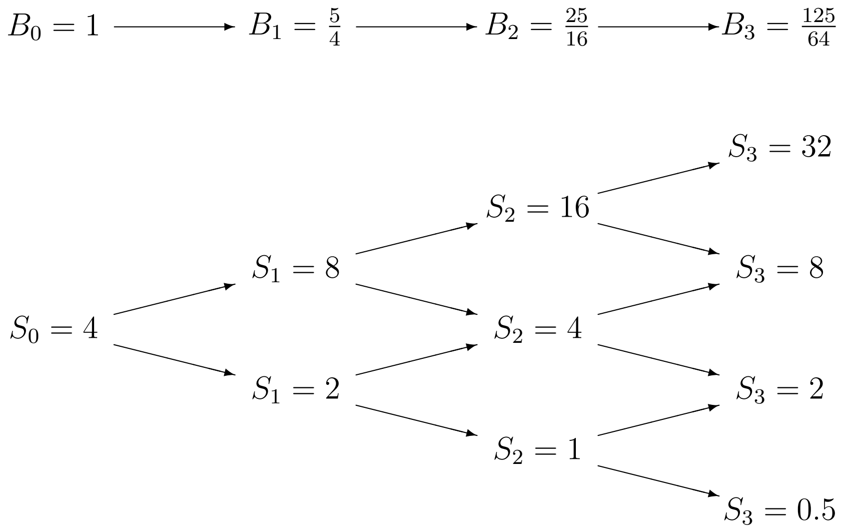
b) $(K - S_t)^+ \leq P_t^a \leq K$,

c) $S_t - K \leq C_t^a - P_t^a \leq S_t - Ke^{-r(T-t)}$.

These are general arbitrage bounds that hold in any market that is free of arbitrage. Thus do not use any particular model structure, like the Black-Scholes model.

Please turn over !

5. Consider the following three period binomial model with $u = 2$ and $d = \frac{1}{2}$, $r = \frac{1}{4}$ and $S_0 = 4$ in discrete time :



Find the time-zero price and optimal exercise policy for the American put that expires at time three and has strike 4.