Mathematical Finance 2: Continuous-Time Models
Exercise sheet 7

May 7, 2013

1. Consider the multi-period Cox-Ross-Rubinstein binomial model with bank account $B_n = (1+r)^n$, $n = 0, \dots, T$, $r \geq 0$, and stock price process S, where S evolves between two consecutive periods as

$$S_{n+1} = S_n Z_{n+1}, \quad n = 0, \dots, T-1, \quad S_0 > 0.$$

Here $(Z_n)_{n=1,\dots,T}$ are i.i.d random variables, taking only the two values u and d for $u > 1+r > d > 0$ with probabilities

$$\mathbb{P}[Z_n = u] = p \quad \text{and} \quad \mathbb{P}[Z_n = d] = 1-p, \quad p \in (0,1).$$

Denote by X^a a path-independent American claim with reward function g, i.e., $X^a_n = g(S_n, n)$. Show that the price of the American claim at time n, given by

$$V_n := \max_{\tau \in \mathcal{T}_{[n,T]}} \mathbb{E}_{\mathbb{P}^*}\left[\left(\frac{1}{1+r} \right)^{\tau - n} X^a_\tau \,\middle|\, \mathcal{F}_n \right]$$

can be recursively computed by

$$\begin{aligned} V_T &= g(S_T, T) \\ V_n &= \max\left(g(S_n, n), \mathbb{E}_{\mathbb{P}^*}\left[\frac{1}{1+r} V_{n+1} \,\middle|\, \mathcal{F}_n \right] \right), \quad n = N-1, \dots, 0, \end{aligned} \tag{1}$$

with

$$\mathbb{E}_{\mathbb{P}^*}\left[\frac{1}{1+r} V_{n+1} \,\middle|\, \mathcal{F}_n \right] = \frac{1}{1+r}\left[p^* V^u_{n+1} + (1-p^*) V^d_{n+1} \right],$$

where $p^* = \frac{1+r-d}{u-d}$ and V^u_{n+1} and V^d_{n+1} denote the values of V defined via (1) at time $n+1$ in the nodes that correspond to the upward and downward movements of the stock price during the time period $[n, n+1]$.

2. Consider the three-period binomial model of Exercise 6.5.

Please turn over!

a) Find the time-zero price and optimal exercise policy (optimal stopping time) for the American Straddle option that expires at time three and has strike 4 (i.e., it has intrinsic value $|S-4|$).

b) Compute the time-zero price of the American call.

c) The value of the American Straddle option (which you computed in part a)) is strictly smaller than the sum of the values of the American put (which you computed in Exercise 6.5) and the call (which you computed in part b), even if $|S-4| = (S-4)^+ + (4-S)^+$; how is this possible ?

3. Consider the Black and Scholes model for the stock price process S and assume that the interest rate r is constant. Let g be a nonnegative continuous reward function such that

$$\mathbb{E}_{\mathbb{P}^*}\left[\sup_{t\in[0,T]} e^{-rt} g(S_t,t)\right] < \infty.$$

For $t \in [0,T]$ and $T > 0$, set

$$J_t = \operatorname{ess}\sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}_{\mathbb{P}^*}\left[e^{-r\tau} g(S_\tau,\tau) \,|\, \mathcal{F}_t\right],$$

where $\mathcal{T}_{[t,T]}$ denotes the set of all stopping times (with respect to the filtration $(\mathcal{F}_t)$ generated by S) which satisfy $t \le \tau \le T$ a.s.

a) Prove that J is a supermartingale.

b) Show that for every $t \in [0,T]$

$$\mathbb{E}_{\mathbb{P}^*}[J_t] = \sup_{\tau\in\mathcal{T}_{[t,T]}} \mathbb{E}_{\mathbb{P}^*}\left[e^{-r\tau} g(S_\tau,\tau)\right].$$

c) A stopping time τ^* is said to be *optimal* if

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-r\tau^*} g(S_{\tau^*},\tau^*)\right] = \sup_{\tau\in\mathcal{T}_{[0,T]}} \mathbb{E}_{\mathbb{P}^*}\left[e^{-r\tau} g(S_\tau,\tau)\right].$$

Prove that a stopping time τ^* is optimal if and only if the following properties are satisfies:

- $J_{\tau^*} = e^{-r\tau^*} g(S_{\tau^*},\tau^*)$ a.s.
- The stopped process J^{τ^*} defined by $J_t^{\tau^*} := J_{\tau^*\wedge t}$, $t \in [0,T]$ is a martingale.