Mathematical Finance 2: Continuous-Time Models
Exercise sheet 8

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1. Consider the Black and Scholes model for the stock price process S and assume that the interest rate r is constant. Let g be a nonnegative continuous reward function such that

$$\mathbb{E}_{\mathbb{P}^*}\left[\sup_{t\in[0,T]} e^{-rt}g(S_t,t)\right] < \infty.$$

Prove that

$$\sup_{\tau\in\mathcal{T}_{[0,T]}} \mathbb{E}_{\mathbb{P}^*}\left[e^{-r\tau}g(S_\tau,\tau)\right] = \inf_{M\in\mathcal{M}_0} \mathbb{E}_{\mathbb{P}^*}\left[\sup_{t\in[0,T]}\left(e^{-rt}g(S_t,t) - M_t\right)\right],$$

where $\mathcal{T}_{[0,T]}$ denotes the set of all stopping times (with respect to the filtration $(\mathcal{F}_t)$ generated by S) which satisfy $0 \le \tau \le T$ a.s. and $\mathcal{M}_0$ is the set of all right continuous martingales $M = (M_t)_{0\le t\le T}$ with $M_0 = 0$.

2. Let W^* be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P}^*)$ and consider the process $X_t = \mu t + W_t^*$ for $\mu \in \mathbb{R}$. Let $m > 0$ and define the stopping time

$$\tau_m = \inf\{t \ge 0 \,|\, X_t = m\},$$

which takes the value ∞ if the level m is never reached. Prove that the Laplace transform of τ_m is given by

$$\mathbb{E}_{\mathbb{P}^*}\left[e^{-\lambda\tau_m}\right] = e^{-m(-\mu+\sqrt{\mu^2+2\lambda})}, \quad \forall\lambda > 0.$$

3. Consider the Black and Scholes model for the stock price process S (with infinite time horizon) and assume that the interest rate $r > 0$ is constant. Consider a perpetual American put, that is, a derivative (with infinite maturity) which pays $K - S_t$ if exercised at time t. Suppose that the owner of the American put sets a level $L < K$ and exercises the put the first time the stock price falls to L. If

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the initial price x satisfies $x \leq L$, the owner exercises the option immediately, otherwise at the stopping time

$$\tau_L = \inf\{t \geq 0 \,|\, S_t = L\},$$

where $\tau_L = \infty$ if the stock prices never reaches L.

a) Show that the value at time 0 (as a function of the initial stock price x) of the perpetual American put under this exercise strategy is given by

$$v_L(x) = \begin{cases} K - x & \text{if } 0 \leq x \leq L, \\ (K-L)\mathbb{E}_{\mathbb{P}^*}\left[e^{-r\tau_L}\right] = (K-L)(\frac{x}{L})^{-\frac{2r}{\sigma^2}} & \text{if } x \geq L. \end{cases}$$

b) Define

$$L_* = \frac{2r}{2r+\sigma^2}K.$$

Show that L_* maximizes the function $L \mapsto v_L(x)$ for every fixed x.

c) Show that v_{L_*} satisfies the so-called *linear complementarity conditions*, that is

i) $v_{L_*}(x) \geq (K-x)^+ \quad$ for all $x \geq 0$,
ii) $rv_{L_*}(x) - rxv'_{L_*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L_*}(x) \geq 0 \quad$ for all $x \geq 0$,
iii) for every $x \geq 0$, equality holds either in i) or in ii).

4. Consider the setting of exercise 3.

a) Prove that

- $e^{-rt}v_{L_*}(S_t)$ is a $\mathbb{P}^*$-supermartingale and
- the stopped process $e^{-r(t\wedge\tau_{L_*})}v_{L_*}(S_{t\wedge\tau_{L_*}})$ is a $\mathbb{P}^*$-martingale.

b) Conclude that the price of the American perpetual put defined via

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}_{\mathbb{P}^*}\left[e^{-r\tau}(K-S_\tau)\right]$$

is given by $v_{L_*}(x)$, where $x = S_0$ and $\mathcal{T}$ denotes the set of all stopping times.

c) Describe the hedging strategy of the seller of the American perpetual put, who has initial capital $V_0 = v_{L_*}(x)$.

5. Consider the setting of Exercise 3 and an American put with strike K and finite maturity T. Define

$$b^*(T-t) = \sup\{x \geq 0 \,|\, P^a(x, T-t) = (K-x)^+\},$$

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where $P^a(x, T-t)$ denotes the price at time t of the American put if the time-t stock price is x, that is,

$$P^a(x, T-t) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{[t,T]}} \mathbb{E}_{\mathbb{P}^*}\left[e^{-r(\tau - t)}(K - S_\tau)^+ \mid S_t = x\right].$$

a) Show that $L_* \leq b^*(t) \leq K$ for all $t \in [0, T]$.

b) Deduce the following bounds for the early exercise premium, that is, the difference between the price of the American and the European put, where the latter is denoted by $P^e(x, t)$ (here t is time to maturity)

$$rK \int_0^t e^{-ru} N\left(\frac{\ln\left(\frac{L_*}{x}\right) - (r - \frac{\sigma^2}{2})u}{\sigma\sqrt{u}}\right) du \leq P^a(x,t) - P^e(x,t)$$
$$\leq rK \int_0^t e^{-ru} N\left(\frac{\ln\left(\frac{K}{x}\right) - (r - \frac{\sigma^2}{2})u}{\sigma\sqrt{u}}\right) du.$$

Hint: You can use the results of Musiela and Rutkowski, Section 5.3.