Mathematical Finance 2: Continuous-Time Models
Exercise sheet 9

June 11, 2013

Solve Exercise 4b,c and 5 of Sheet 8 and the following new exercises.

1. The holder of a *forward-start option* receives at time T_0 an option with maturity $T > T_0$ and strike KS_{T_0} for some $K > 0$. Compute the price of a *forward-start call* at time $t = 0$ in the Black-Scholes model with constant dividend yield $\varkappa$.

2. Consider a *gap call* with payoff

$$H(S_T) = (S_T - L)1_{\{S_T > K\}},$$

where $K, L \geq 0$. Draw the payoff function. For which values of K and L does it take negative values? Find the price and the hedging strategy in the Black-Scholes model by splitting the option in digital options.

3. Consider the Black-Scholes model with parameters μ, σ and $r \geq 0$. Let

$$H := S_T - \inf_{0 \leq u \leq T} S_u$$

be the payoff function of a so-called European *floating strike lookback call.*

a) Define $M_t = \inf_{0 \leq u \leq t} S_u,\ t \in [0, T]$ and show that the two dimensional process (S, M) is Markov. That is, show that for every $0 \leq s \leq t \leq T$ and bounded function $f : \mathbb{R}^2 \to \mathbb{R}$ there exists a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\mathbb{E}\left[f(S_t, M_t) | \mathcal{F}_s\right] = g(S_s, M_s).$$

Conclude that the price of the floating strike lookback call at time t is given by

$$V_t = v(t, S_t, M_t)$$

for some function $v(t, x, m)$.

Please turn over!

b) Apply Itô's formula to $e^{-rt}v(t, S_t, M_t)$ to find a PDE for the price the floating strike lookback call. Determine its domain and the boundary conditions. *Hint:* The process M_t is decreasing and continuous, but its absolutely continuous part is a.s. constant.

c) Show that the price of an American lookback call is the same as the European counterpart.

4. Consider the Black-Scholes model and a zero-strike Asian call whose payoff at time T is

$$H = \frac{1}{T} \int_0^T S_u du.$$

a) Suppose at time t we have $S_t = x \geq 0$ and $Y_t = \int_0^t S_u du = y \geq 0$. Use the fact that $(e^{-ru} S_u)_{u \in [0,T]}$ is martingale under $\mathbb{P}^*$ to compute

$$e^{-r(T-t)} \mathbb{E}_{\mathbb{P}^*} \left[\frac{1}{T} \int_0^T S_u du | \mathcal{F}_t \right]$$

and denote this by $v(t, x, y)$.

b) Determine explicitly the process $\Delta_t = v_x(t, S_t, Y_t)$ and observe that it is not random.

c) Use Itô's formula to show that if you begin with initial capital $X_0 = v(0, S_0, 0)$ and at each time you hold Δ_t shares of S, investing or borrowing at the interest rate r in order to do this, then at time T the value of your portfolio will be

$$X_T = \frac{1}{T} \int_0^T S_u du.$$