Mathematical Finance 2 : Continuous-Time Models
Exercise sheet 10

June 13, 2013

1. The aim of this exercise is to prove Lee's moment formula, as stated in Proposition 7.1.2 in Musiela/Rutkowski. Let the interest rate be equal to 0, denote by S_0 the initial stock price, and define the log-moneyness x related to strike by

$$x = \log\left(\frac{K}{S_0}\right),$$

so that $K(x) = S_0 e^x$ is the strike at log-moneyness x. Black-Scholes implied volatility at log-moneyness x is defined as the unique solution $\hat{\sigma}_0(x)$ of the equation $C_0(T, K(x)) = c(S_0, T, K(x), \hat{\sigma}_0(x))$, where

$$c(S_0, T, K(x), \sigma) = S_0 N(d_+) - K(x) N(d_-),$$

with

$$d_\pm = \frac{-x}{\sigma\sqrt{T}} \pm \frac{\sigma\sqrt{T}}{2}.$$

Let $\beta > 0$ and let us denote by $f_-(\beta)$

$$f_-(\beta) = \frac{1}{\beta} + \frac{\beta}{4} - 1.$$

Furthermore, define

$$\widetilde{p} = \sup\left\{p > 0 \,|\, \mathbb{E}_{\mathbb{P}^*}\left[S_T^{p+1}\right] < \infty\right\}$$

and

$$\widetilde{\beta} = \limsup_{x\to\infty} \frac{T\hat{\sigma}_0^2(x)}{x}.$$

Lee's moment formula states the following relation between $\widetilde{p}$ and $\widetilde{\beta}$:

$$\widetilde{p} = \frac{f_-(\widetilde{\beta})}{2},$$

where $1/0 := \infty$ if $\widetilde{\beta} = 0$.

Please turn over !

a) Prove for $\beta \in (0,2)$

$$\lim_{x\to\infty} \frac{e^{-cx}}{c\left(S_0, T, K(x), \sqrt{\frac{\beta|x|}{T}}\right)} = \begin{cases} 0 & \text{if } c > \frac{f_-(\beta)}{2} \\ \infty & \text{if } c \le \frac{f_-(\beta)}{2} \end{cases}.$$

b) Use this and Proposition 7.1.1 to show that

$$\lim_{x\to\infty} \frac{c\left(S_0, T, K(x), \hat{\sigma}_0(x)\right)}{c\left(S_0, T, K(x), \sqrt{\frac{\beta|x|}{T}}\right)} = 0$$

for every $\beta \in (0,2)$ satisfying $\frac{f_-(\beta)}{2} < \widetilde{p}$ and conclude that $\widetilde{p} \le \frac{f_-(\widetilde{\beta})}{2}$.

c) Let f be a twice-differentiable function and let $\varkappa \ge 0$. Prove that

$$f(S_T) = f(\varkappa) + f'(\varkappa)((S_T - \varkappa)^+ - (\varkappa - S_T)^+) + \int_0^{\varkappa} f''(K)(K - S_T)^+ dK + \int_{\varkappa}^{\infty} f''(K)(S_T - K)^+ dK$$

and conclude that

$$\mathbb{E}_{\mathbb{P}^*}[S_t^{p+1}] = \mathbb{E}_{\mathbb{P}^*}\left[\int_0^\infty (p+1)pK^{p-1}(S_T - K)^+ dK\right]. \tag{1}$$

d) Use (1) to show that for all $0 < p < \frac{f_-(\widetilde{\beta})}{2}$, $\mathbb{E}_{\mathbb{P}^*}[S_t^{p+1}] < \infty$. Conclude using b) that $\widetilde{p} = \frac{f_-(\widetilde{\beta})}{2}$.

2. Let T^* denote a finite time horizon and consider the following model for the stock price under $\mathbb{P}^*$:

$$dS_t = S_t(rdt + \sigma_t dW_t^*), \quad S_0 > 0,$$

where $r \ge 0$ denotes the interest rate and σ is a progressively measurable process satisfying

$$\mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] < \infty, \quad 0 \le T \le T^*.$$

Assume in addition that the random variable S_T admits a density for every $0 < T \le T^*$. Define

$$c(K,T) = e^{-rT}\mathbb{E}_{\mathbb{P}^*}\left[(S_T - K)^+\right]$$

and assume that it is of class $C^{1,2}(\mathbb{R} \times [0,T^*], \mathbb{R})$. Let

$$\sigma^2(K,T) := \mathbb{E}_{\mathbb{P}^*}\left[\sigma_T^2 | S_T = K\right]$$

and show that $c(K,T)$ satisfies

$$-c_T(K,T) + \frac{1}{2}\sigma^2(K,T)K^2 c_{KK}(K,T) - rKc_K(K,T) = 0$$

by mimicing the proof of Proposition 7.3.1 in Musiela/Rutkowski.