Mathematical Finance 2 : Continuous-Time Models
Exercise sheet 11

June 20, 2013

Please prepare Exercise 1c,d and 2 of Sheet 10 and the following new exercises.

1. Consider a stock price model based on a 2-dimensional Brownian motion (W, W') in its own filtration, where the (one-dimensional) stock price is given by

$$dS_t = \mu_t \, dt + \sigma_t \, dW_t.$$

The coefficients $\mu_t = \mu(t, S_t, Y_t)$ and $\sigma_t = \sigma(t, S_t, Y_t)$ are determined by continuous functions depending on the autonomous diffusion Y which follows

$$dY_t = b(t, Y_t) \, dt + a(t, Y_t) \, dB_t, \quad Y_0 = y_0.$$

Here B is a correlated Brownian motion defined by $B_t = \varrho W_t + \sqrt{1-\varrho^2} W'_t$ for some constant $\varrho \in (0,1)$. We assume that the function μ/σ is uniformly bounded.

a) Show that $d\langle W, B\rangle_t = \varrho \, dt$ and $d\langle S, Y\rangle_t = a(t, Y_t)\sigma(t, S_t, Y_t)\varrho \, dt$.

b) What is the general form of the density process of an equivalent local martingale measure $\mathbb{P}^*$ for S?

c) Find the dynamics of S and Y under such a measure. That is, give stochastic differential equations for these processes involving only Brownian motions under $\mathbb{P}^*$, not under $\mathbb{P}$.

2. Consider a multidimensional Itô process model with finite time horizon $T > 0$, where for any $i = 1, \dots, d$ the dynamics of the i^{th} stock price process are given by

$$dS_t^i = S_t^i \mu^i dt + S_t^i \sum_{j=1}^{n} \sigma_t^{ij} dW_t^j, \quad S_0^i = s^i > 0.$$

Here, $W = (W^1, \dots, W^n)$ denotes a standard n-dimensional Brownian motion on a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}_t)_{0\le t\le T}$ is the filtration generated by W, satisfying the usual conditions. The coefficients σ and μ

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are assumed to be bounded predictable processes with values in $\mathbb{R}^{d\times n}$ and $\mathbb{R}^d$ respectively. The bank account is modeled by

$$dB_t = B_t r_t dt, \quad B_0 = 1.$$

We suppose furthermore that there exists an equivalent local martingale measure $\mathbb{P}^*$ for the discounted price process. Assume now that the number of stocks d is equal to the dimension n of the underlying Brownian motion and that the volatility matrix $\sigma = (\sigma^{ij})$ is $d\mathbb{P}\otimes dt$-a.s. nonsingular. Prove that these conditions imply the following :

a) The market price of risk λ is $d\mathbb{P} \otimes dt$-a.s. unique.

b) The equivalent local martingale measure $\mathbb{P}^*$ is unique.

c) Every discounted payoff $H \in L^\infty(\mathcal{F}_T)$ is attainable (with a self-financing admissible strategy). *Hint :* Consider the $\mathbb{P}^*$-martingale with final value H and use the martingale representation theorem under $\mathbb{P}$.

3. Let $T > 0$ denote a finite time horizon and consider a stochastic process X, which is *not* assumed to be the price process of a traded asset, with dynamics given by

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where μ and σ are some deterministic functions and W a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let B denote the bank account with dynamics

$$dB_t = rB_t dt,$$

where r is as usual the deterministic interest rate. We consider a given contingent claim, written in terms of X. More specifically we define the claim by

$$H = \varphi(X_T)$$

for some given deterministic function φ and assume that the market price of the claim at time $t \leq T$, denoted by $\Pi_t(H)$, is given by

$$\Pi_t(H) = G(t, X_t),$$

where G is a smooth function satisfying $G(T, X_T) = \varphi(X_T)$. Moreover, assume that the volatility function $\sigma_G(t, X_t)$ of $G(t, X_t)$ is nonzero. Show that every claim of the form $\psi(X_T)$ for some deterministic function ψ can be replicated by a portfolio based on B and $\Pi(H)$.