Name:

Mat.Nr.:

Bitte keinen Rotstift verwenden!

Finanzmathematik 2: Modelle in stetiger Zeit (Vorlesungsprüfung)
7. März 2014
Privatdoz. Dr. Stefan Gerhold

90 Minuten

Unterlagen: ein handbeschriebener A4-Zettel sowie ein nichtprogrammierbarer Taschenrechner sind erlaubt

Anmeldung zur mündlichen Prüfung via TISS möglich. Wenn zu wenig Prüfungstermine online sind, bitte den Vortragenden Stefan Gerhold kontaktieren.

Bsp.	Max.	Punkte
1	12	
2	8	
3	8	
$\sum$	28	

Schriftlich:

AssistentIn:

Mündlich:

Gesamtnote:

1. Fix a time horizon $T \in (0, \infty)$ and a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which there is a Brownian motion $(W_t)_{0\le t\le T}$. We take as filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the one generated by W and augmented by the $\mathbb{P}$-nullsets in $\sigma(W_s; s \le T)$. Consider the Black Scholes model, where the bank account satisfies $B \equiv 1$, i.e. the interest rate $r \equiv 0$, and the risky asset price is given by (12 Pkt.)

$$dS_t = S_t(\mu dt + \sigma dW_t), \quad S_0 = 1,$$

where $\mu \in \mathbb{R}$ and $\sigma > 0$. Moreover, $\mathbb{P}^* \sim \mathbb{P}$ denotes the unique equivalent martingale measure for the discounted price process $S = \frac{S}{B}$.

(a) Let H be a nonnegative $\mathcal{F}_T$-measurable payoff due at time T.

(i) Construct a probability measure $\widehat{\mathbb{P}} \sim \mathbb{P}^*$ such that

$$E_{\mathbb{P}^*}[H] = E_{\widehat{\mathbb{P}}}\left[\frac{H}{S_T}\right].$$

Specify in particular the candidate density process $(Z)_{0\le t\le T}$ and show that it satisfies all necessary properties such that

$$\frac{d\widehat{\mathbb{P}}}{d\mathbb{P}^*} := Z_T \tag{1}$$

defines an equivalent probability measure $\widehat{\mathbb{P}} \sim \mathbb{P}^*$.

(ii) Show that

$$\widehat{W}_t := W_t^* - \sigma t \tag{2}$$

is a $\widehat{\mathbb{P}}$-Brownian motion, where W^* denotes a $\mathbb{P}^*$-Brownian motion.

(iii) Use Bayes' formula to show that $\frac{1}{S}$ is a $\widehat{\mathbb{P}}$-martingale.

(b) Consider the process

$$\widehat{S}_t = \exp\left(-\sigma\widehat{W}_t - \frac{1}{2}\sigma^2 t\right), \tag{3}$$

where $\widehat{W}$ is a $\widehat{\mathbb{P}}$-Brownian motion, as specified in (2), and $\widehat{\mathbb{P}}$ denotes the measure defined in (1).

(i) Derive the SDE satisfied by $\widehat{S}$ under $\widehat{\mathbb{P}}$.

(ii) What is the relation between $\widehat{S}$ and $\frac{1}{S}$?

(c) Consider a lookback call option with floating strike, whose payoff at time T is given by

$$H = \left(S_T - \alpha \min_{0\le t\le T} S_t\right)^+, \quad \alpha \ge 1. \tag{4}$$

Show that its price at time 0 can be expressed by

$$\alpha E_{\widehat{\mathbb{P}}}\left[\left(\frac{1}{\alpha} - \min_{0\le t\le T} \widehat{S}_t\right)^+\right],$$

where $\widehat{S}_t$ is given by (3) and $\widehat{\mathbb{P}}$ denotes the measure defined in (1).
Hint: Use the reflection principle for a Brownian motion with drift: If $X_t = bt + c\widehat{W}_t$, $b, c \in \mathbb{R}$ and $\widehat{W}$ a $\widehat{\mathbb{P}}$-Brownian motion, then we have for all $x \in \mathbb{R}$

$$\widehat{\mathbb{P}}\left[\max_{0\leq t\leq T} X_t - X_T \leq x\right] = \widehat{\mathbb{P}}\left[-\min_{0\leq t\leq T} X_t \leq x\right].$$

This means that $\max_{0\leq t\leq T} X_t - X_T$ and $-\min_{0\leq t\leq T} X_t$ have the same law.

(d) Let $\alpha = 1$ in (4). Show that the price of an American lookback call is the same as the European counterpart.

2. Fix a time horizon $T \in (0, \infty)$ and a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ equipped with a standard Brownian motion $(W_t)_{0\leq t\leq T}$. Consider the following Itô-process model for the stock price (8 Pkt.)

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t), \quad S_0 > 0,$$

where μ and $\sigma > 0$ are bounded predictable processes. Moreover, consider a continuously monitored variance swap contract with payoff

$$V_T = \frac{1}{T}\int_0^T \sigma_t^2 dt.$$

(a) Prove that

$$V_T = \frac{2}{T}\left(\int_0^T \frac{1}{S_t} dS_t - \ln\left(\frac{S_T}{S_0}\right)\right).$$

(b) Show that for $\kappa \geq 0$

$$-\ln\left(\frac{S_T}{\kappa}\right) = -\frac{S_T - \kappa}{\kappa} + \int_0^\kappa \frac{1}{K^2}(K - S_T)^+ dK + \int_\kappa^\infty \frac{1}{K^2}(S_T - K)^+ dK.$$

Hint: For a twice-differentiable function and $\kappa \geq 0$, the following formula holds:

$$\begin{aligned} f(S_T) &= f(\kappa) + f'(\kappa)(S_T - \kappa) \\ &\quad + \int_0^\kappa f''(K)(K - S_T)^+ dK + \int_\kappa^\infty f''(K)(S_T - K)^+ dK. \end{aligned}$$

(c) Let $r \geq 0$ denote the deterministic constant interest rate and let the call and put prices with maturity T and strike K be given by

$$C(K) = \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}(S_T - K)^+\right], \quad P(K) = \mathbb{E}_{\mathbb{P}^*}\left[e^{-rT}(K - S_T)^+\right],$$

where $\mathbb{P}^* \sim \mathbb{P}$ denotes some equivalent martingale measure for the discounted stock price $(e^{-rt}S_t)_{0\leq t\leq T}$. Show that the price of the variance swap defined via $\mathbb{E}_{\mathbb{P}^*}[e^{-rT}V_T]$ is given by

$$\frac{2}{T}\left(\int_0^{F_T} \frac{1}{K^2} P(K) dK + \int_{F_T}^\infty \frac{1}{K^2} C(K) dK\right),$$

where $F_T = e^{rT} S_0$.

3. The holder of a *forward-start option* receives at time T_0 an option with maturity $T > T_0$ and strike KS_{T_0} for some $K > 0$. (8 Pkt.)

   (a) Compute the price of a *forward-start call* at time $t = 0$ in the Black-Scholes model with constant dividend yield κ.

   (b) Determine the limits of the option price as $T_0 \to 0$ and $T_0 \to \infty$ as well as $\sigma \to 0$ and $\sigma \to \infty$.