KORRELATIONSKOEFFIZIENT f. BIVARIATE GUMBELVERTEILUNG

(X,Y) hat cdf $F(x,y) = e^{-(e^{-x/\rho}+e^{-y/\rho})^{\rho}}$ $\quad \rho \in (0,1]$

Gumbel-Margins: X hat Randverteilung mit cdf $\lim_{y\to\infty} F(x,y) = e^{-e^{-x}}$

analog f. Y

Wissen: $E(X) = E(Y) = \gamma$ (Euler-Mascheroni Konstante)

$Var(X) = Var(Y) = \pi^2/6$

$$Corr(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \frac{E(XY)-\gamma^2}{\pi^2/6}$$

$$E(XY) = \int_{\mathbb{R}^2} xy\, f(x,y)\, d(x,y)$$

mit $f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y) = \frac{\partial}{\partial y}\left(e^{-(e^{-x/\rho}+e^{-y/\rho})^{\rho}} (-\rho)\left(e^{-x/\rho}+e^{-y/\rho}\right)^{\rho-1} e^{-x/\rho}\left(-\tfrac{1}{\rho}\right)\right)$

$$= \frac{\partial}{\partial y}\left[F(x,y)\left(e^{-x/\rho}+e^{-y/\rho}\right)^{\rho-1} e^{-x/\rho}\right]$$

$$= F(x,y)\left(e^{-x/\rho}+e^{-y/\rho}\right)^{\rho-1} e^{-y/\rho}\left(e^{-x/\rho}+e^{-y/\rho}\right)^{\rho-1} e^{-x/\rho}$$
$$+ F(x,y)(\rho-1)\left(e^{-x/\rho}+e^{-y/\rho}\right)^{\rho-2} e^{-x/\rho} e^{-y/\rho}\left(-\tfrac{1}{\rho}\right)$$

$$= F(x,y)\, e^{-(x+y)/\rho}\left(e^{-x/\rho}+e^{-y/\rho}\right)^{\rho-2}\left[\left(e^{-x/\rho}+e^{-y/\rho}\right)^{\rho} + \left(\tfrac{1}{\rho}-1\right)\right]$$

$$\Rightarrow \quad E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, e^{-(e^{-x/\rho}+e^{-y/\rho})^{\rho}}\, e^{-(x+y)/\rho}\, (e^{-x/\rho}+e^{-y/\rho})^{\rho-2} \times$$

$$\times \left[(e^{-x/\rho}+e^{-y/\rho})^{\rho} + (1/\rho - 1)\right] dx\, dy$$

$$= \left[\begin{array}{l} a := e^{-x/\rho} \\ b := e^{-y/\rho} \end{array} \Rightarrow \begin{array}{l} x = -\rho \ln a \\ y = -\rho \ln b \end{array} \quad \text{und} \quad da\, db = \begin{vmatrix} -\frac{1}{\rho} e^{-x/\rho} & 0 \\ 0 & -\frac{1}{\rho} e^{-y/\rho} \end{vmatrix} dx\, dy \right]$$

$$= \frac{1}{\rho^2}\, e^{-(x+y)/\rho}\, dx\, dy$$

$$= \int_0^{\infty}\int_0^{\infty} \rho^4 \ln a \ln b\, e^{-(a+b)^{\rho}}\, (a+b)^{\rho-2} \left[(a+b)^{\rho} + (1/\rho - 1)\right] da\, db$$

$$= \left[\begin{array}{l} u := a+b \\ v := a \end{array} \Rightarrow \begin{array}{l} a = v \\ b = u-v \end{array} \quad \text{und} \quad dv\, du = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} da\, db = da\, db \right]$$

$$= \rho^4 \int_0^{\infty}\int_0^{u} \ln(v) \ln(u-v)\, e^{-u^{\rho}}\, u^{\rho-2} \left[u^{\rho} + (1/\rho - 1)\right] dv\, du$$

$$= \rho^4 \int_0^{\infty} e^{-u^{\rho}}\, u^{\rho-2} \left[u^{\rho} + (1/\rho - 1)\right] \int_0^{u} \ln v \ln(u-v)\, dv \quad du$$

verwendete Integrale:

$$\int_0^u \ln v \,\ln(u-v)\,dv = \left(2-\tfrac{\pi^2}{6}\right)u + u\ln u\,(\ln u - 2)$$

$$-\int_0^\infty \ln x \, e^{-x}\,dx = \gamma \qquad \text{(Euler - Mascheroni Konstante)}$$

$$\int_0^\infty (\ln x)^2 e^{-x}\,dx = \begin{bmatrix} z := -\ln x \\ x = e^{-z} \\ dx = -e^{-z}dz \end{bmatrix} = \int_\infty^{-\infty} z^2 e^{-e^{-z}} - e^{-z}\,dz$$

$$= \int_{-\infty}^{\infty} z^2 e^{-z} e^{-e^{-z}}\,dz = E(Z^2) \quad \text{f. } Z \sim \text{Gumbel}$$

$$= E(Z)^2 + Var(Z) = \gamma^2 + \tfrac{\pi^2}{6}$$

$$\Rightarrow \; E(XY) = \rho^4 \int_0^\infty e^{-u^\rho} u^{\rho-2} \left[u^\rho + \left(\tfrac{1}{\rho}-1\right)\right]\left[\left(2-\tfrac{\pi^2}{6}\right)u + u(\ln u)^2 - 2u\ln u\right] du$$

$$= \rho^4\left(2-\tfrac{\pi^2}{6}\right) \underbrace{\int_0^\infty e^{-u^\rho} u^{2\rho-1}\,du}_{(a)} + (\rho^3-\rho^4)\left(2-\tfrac{\pi^2}{6}\right)\underbrace{\int_0^\infty e^{-u^\rho} u^{\rho-1}\,du}_{(b)}$$

$$+ \rho^4 \underbrace{\int_0^\infty e^{-u^\rho} u^{2\rho-1}(\ln u)^2\,du}_{(c)} + (\rho^3-\rho^4)\underbrace{\int_0^\infty e^{-u^\rho} u^{\rho-1}(\ln u)^2\,du}_{(d)}$$

$$- 2\rho^4 \underbrace{\int_0^\infty e^{-u^\rho} u^{2\rho-1}\ln u\,du}_{(e)} - 2(\rho^3-\rho^4)\underbrace{\int_0^\infty e^{-u^\rho} u^{\rho-1}\ln u\,du}_{(f)}$$

Substitution: $w = u^\rho$ Grenzen: $u = +\infty \Rightarrow w = +\infty$

$u = 0 \Rightarrow w = 0$

$\frac{1}{\rho}\,dw = u^{\rho-1}\,du$

$\ln u = \frac{1}{\rho}\ln u^\rho = \frac{1}{\rho}\ln w$

$$(a) = \frac{1}{\rho}\int_0^\infty e^{-w}\, w\, dw = \underset{\text{f. } W\sim \exp(\lambda=1)}{\frac{1}{\rho}E(W)} = \frac{1}{\rho}$$

$$(b) = \frac{1}{\rho}\int_0^\infty e^{-w}\, dw = \frac{1}{\rho}$$

$$(c) = \frac{1}{\rho^3}\int_0^\infty e^{-w}\, w(\ln w)^2\, dw = \text{(partielle Integration)}$$

$$= \frac{1}{\rho^3}\left[\underbrace{-w(\ln w)^2 e^{-w}\Big]_0^\infty}_{=0} + \int_0^\infty [2\ln w + (\ln w)^2]\, e^{-w}\, dw\right]$$

$$= \frac{1}{\rho^3}\left[-2\gamma + \left(\frac{\pi^2}{6} + \gamma^2\right)\right]$$

$$(d) = \frac{1}{\rho^3}\int_0^\infty (\ln w)^2 e^{-w}\, dw = \frac{1}{\rho^3}\left(\frac{\pi^2}{6} + \gamma^2\right)$$

$$(e) = \frac{1}{\rho^2}\int_0^\infty e^{-w}\, w \ln w\, dw = \text{(partielle Integration)}$$

$$= \frac{1}{\rho^2}\left[\underbrace{-w\ln w\, e^{-w}\Big]_0^\infty}_{=0} + \int_0^\infty (\ln w + 1)\, e^{-w}\, dw\right]$$

$$= \frac{1}{\rho^2}\left[-\gamma + 1\right]$$

$$(f) = \frac{1}{\rho^2}\int_0^\infty e^{-w} \ln w\, dw = -\frac{1}{\rho^2}\gamma$$

$$\Rightarrow \quad E(XY) = \rho^3\left(2-\tfrac{\pi^2}{6}\right) + (\rho^2-\rho^3)\left(2-\tfrac{\pi^2}{6}\right) + \rho\left[\tfrac{\pi^2}{6}+\gamma^2-2\gamma\right]$$

$$+ (1-\rho)\left[\tfrac{\pi^2}{6}+\gamma^2\right] - 2\rho^2(1-\gamma) + 2(\rho-\rho^2)\gamma$$

$$= \rho^2\left(2-\tfrac{\pi^2}{6}\right) - 2\rho\gamma + \left(\tfrac{\pi^2}{6}+\gamma^2\right) - 2\rho^2 + 2\rho\gamma$$

$$= \gamma^2 + \tfrac{\pi^2}{6}(1-\rho^2)$$

$$\Rightarrow \quad \mathrm{Corr}(X,Y) = \frac{\gamma^2 + \frac{\pi^2}{6}(1-\rho^2) - \gamma^2}{\pi^2/6} = 1-\rho^2$$