$\boxed{11}$ Die Log-Likelihood Funktion im BPM ist strikt konkav falls die Regressormatrix vollen Spaltenrang hat.

Bew: $\ell_n(\beta) = \sum_{i=1}^{n} y_i\, g(x_i\beta) + (1-y_i)\, h(x_i\beta)$

mit $g(z) = \ln \Phi(z)$ und $h(z) = \ln(1-\Phi(z))$

$$\Rightarrow \frac{\partial^2}{\partial\beta\partial\beta'} \ell_n(\beta) = \sum_{i=1}^{n} \left[y_i\, g''(x_i\beta) + (1-y_i)\, h''(x_i\beta) \right] x_i' x_i$$

$$g(z) = \ln \Phi(z)$$

$$g'(z) = \frac{1}{\Phi(z)} \varphi(z)$$

$$g''(z) = \frac{-z\varphi(z)\Phi(z) - \varphi^2(z)}{\Phi^2(z)} = -\varphi(z) \frac{z\Phi(z) + \varphi(z)}{\Phi^2(z)} < 0 \quad \forall z \in \mathbb{R}$$

$$\Longleftrightarrow \quad \bar{g}(z) := z\,\Phi(z) + \varphi(z) > 0 \quad \forall z \in \mathbb{R}$$

Es gilt: $\bar{g}'(z) = \Phi(z) + z\varphi(z) - z\varphi(z) = \Phi(z) > 0 \quad \forall z \in \mathbb{R}$

d.h. $\bar{g}(z)$ ist strikt monoton wachsend in z

Außerdem ist $\lim_{z \to -\infty} \bar{g}(z) = \lim_{z\to-\infty} \frac{\Phi(z)}{\frac{1}{z}} \underset{\text{L'Hospital}}{\overset{\uparrow}{=}} \lim_{z\to-\infty} \frac{\varphi(z)}{-\frac{1}{z^2}} = \lim_{z\to-\infty} -z^2\varphi(z) = 0$

Es muss also $\bar{g}(z) > 0 \quad \forall z \in \mathbb{R}$ und somit $\underline{g''(z) < 0 \quad \forall z \in \mathbb{R}}$ gelten

Analog erhält man: $h''(z) < 0 \quad \forall z \in \mathbb{R}$

Es gilt also $\gamma_i := y_i\, g''(x_i\beta) + (1-y_i)\, h''(x_i\beta) < 0$ und

$$H_\beta := \frac{\partial^2}{\partial\beta\,\partial\beta'}\, l_n(\beta) = \sum_{i=1}^{n} \gamma_i\, x_i' x_i \leq 0$$

Bleibt noch z.z.: $\alpha \in \mathbb{R}^k$ mit $\alpha' H_\beta\, \alpha = 0 \Rightarrow \alpha = 0$

$$\alpha' H_\beta\, \alpha = \sum_{i=1}^{n} \gamma_i\, \alpha' x_i' x_i\, \alpha = 0 \Rightarrow \alpha' x_i' x_i\, \alpha = 0 \quad i = 1,..,n \text{ da } \gamma_i > 0$$

$$\Rightarrow 0 = \sum_i \alpha' x_i' x_i\, \alpha = \alpha' \Big(\sum_i x_i' x_i\Big) \alpha = \alpha' X' X \alpha$$

$\Rightarrow \alpha = 0$ da $rk(X) = 0$.