$\boxed{5}$ $\quad y = X\beta_0 + u \qquad u \sim N(0, \sigma_0^2 I_n), \qquad \theta = (\beta', \sigma^2)' \in \mathbb{R}^{k+1}$

Log-Likelihood Funktion $l_n(\theta) = -\frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\|y - X\beta\|^2$

$$\frac{\partial}{\partial\theta} l_n(\theta) = \begin{pmatrix} -\frac{1}{\sigma^2}(X'X\beta - 2X'y) \\ -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\|y - X\beta\|^2 \end{pmatrix}$$

$$\frac{\partial^2}{\partial\theta\partial\theta'} l_n(\theta) = \begin{pmatrix} -\frac{1}{\sigma^2}X'X & \frac{1}{\sigma^4}(X'X\beta - X'y) \\ \frac{1}{\sigma^4}(X'X\beta - X'y) & \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\|y - X\beta\|^2 \end{pmatrix}$$

$$\Rightarrow E_{\theta_0}\left(-\frac{\partial^2}{\partial\theta\partial\theta'} l_n(\theta_0)\right) = \begin{pmatrix} \frac{1}{\sigma_0^2}X'X & E_{\theta_0}(X'y - X'X\beta_0) \\ \underbrace{E_{\theta_0}(X'y - X'X\beta_0)}_{= X'X\beta_0 - X'X\beta_0 = 0} & \underbrace{\frac{n}{2\sigma_0^4} - \frac{1}{\sigma_0^6}E\|y - X\beta_0\|^2}_{= \frac{n}{2\sigma_0^4} - \frac{1}{\sigma_0^6}E\|u\|^2 = \frac{n}{2\sigma_0^4} - \frac{1}{\sigma_0^6}n\sigma_0^2 = \frac{n}{2\sigma_0^4}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sigma_0^2}X'X & 0 \\ 0 & \frac{n}{2\sigma_0^4} \end{pmatrix} = I(\theta_0)$$

Restriktion $\quad R\beta = r \quad g(\theta) = R\beta - r = 0 \qquad R \in \mathbb{R}^{q \times k}$

$$\Rightarrow \frac{\partial}{\partial\theta} g(\theta) = (R \;\; 0) = \tilde{R} \in \mathbb{R}^{q \times (k+1)}$$

Wald-Statistik $\quad T_W = g(\hat\theta_{ML})' \left(\frac{\partial}{\partial\theta} g(\hat\theta_{ML})' \, I(\hat\theta_{ML})^{-1} \frac{\partial}{\partial\theta'} g(\hat\theta_{ML})\right)^{-1} g(\hat\theta_{ML})$

$$= (R\hat\beta - r)' \left((R \;\; 0)\begin{pmatrix} \frac{1}{\hat\sigma_{ML}^2}X'X & 0 \\ 0 & \frac{n}{2\hat\sigma_{ML}^4} \end{pmatrix}^{-1} \begin{pmatrix} R \\ 0 \end{pmatrix}\right)^{-1} (R\hat\beta - r)$$

$$= (R\hat\beta - r)' \left((R \;\; 0)\begin{pmatrix} \hat\sigma_{ML}^2(X'X)^{-1} & 0 \\ 0 & \frac{2\hat\sigma_{ML}^4}{n} \end{pmatrix} \begin{pmatrix} R \\ 0 \end{pmatrix}\right)^{-1} (R\hat\beta - r)$$

$$= \frac{(R\hat{\beta}-r)'\,(R(X'X)^{-1}R')^{-1}(R\hat{\beta}-r)}{\hat{\sigma}^2_{ML}}$$

da $\hat{\sigma}^2_{ML} = \frac{1}{n}\hat{u}'\hat{u} = \frac{n-k}{n}\hat{\sigma}^2$ gilt:

$$T_W = \frac{(R\hat{\beta}-r)'\,(R(X'X)^{-1}R')^{-1}(R\hat{\beta}-r)}{\frac{n-k}{n}\hat{\sigma}^2}$$

$$= \underbrace{\frac{q\cdot n}{n-k}}_{\xrightarrow{n\to\infty} q} \underbrace{\frac{(R\hat{\beta}-r)'(R(X'X)^{-1}R')^{-1}(R\hat{\beta}-r)}{\hat{\sigma}^2}\cdot\frac{1}{q}}_{= \text{F-Statistik} \;\sim\; F_{q,\,n-k}}$$

übrigens gilt: $F_{q,\,n-k} \xrightarrow[n\to\infty]{} \chi^2_q$